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The convex core of quasifuchsian manifolds with particles

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Abstract

We consider quasifuchsian manifolds with "particles", i.e., cone singularities of fixed angle less than π going from one connected component of the boundary at infinity to the other. Each connected component of the boundary at infinity is then endowed with a conformal structure marked by the endpoints of the particles. We prove that this defines a homeomorphism from the space of quasifuchsian metrics with n particles (of fixed angle) and the product of two copies of the Teichmüller space of a surface with n marked points. This is analoguous to the Bers theorem in the non-singular case.

Quasifuchsian manifolds with particles also have a convex core. Its boundary has a hyperbolic induced metric, with cone singularities at the intersection with the particles, and is pleated along a measured geodesic lamination. We prove that any two hyperbolic metrics with cone singularities (of prescribed angle) can be obtained, and also that any two measured bending laminations, satisfying some obviously necessary conditions, can be obtained, as in [BO04] in the non-singular case.

Résumé

On considère des variétés quasifuchsiennes "à particules", c'est-à-dire ayant des singularités coniques d'angle fixé inférieur à π allant d'une composante connexe à l'infini à l'autre. Chaque composante connexe du bord à l'infini est alors muni d'une structure conforme marquée par les extrémités des particules. On montre que ceci définit un homéomorphisme de l'espace des métriques quasifuchsiennes à n particules (d'angle fixé) vers le produit de deux copies de l'espace de Teichmüller d'une surface à n points marqués. C'est l'analogue du théorème de Bers dans le cas non-singulier.

Les variétés quasifuchsiennes à particules ont aussi un coeur convexe. Son bord a une métrique induite hyperbolique, avec des singularités coniques aux intersections avec les particules, et est plissé le long d'une lamination géodésique mesurée. On montre que toute paire de métriques hyperboliques à singularités coniques (d'angle prescrit) peut être obtenu, et aussi que toute paire de laminations de plissages, satisfaisant des conditions clairement nécessaires, peut être obtenus, comme dans le cas non-singulier [BO04].

Contents

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1 Introduction

1.1 Convex co-compact manifolds with particles

Quasifuchsian manifolds. A quasifuchsian manifold is a complete hyperbolic manifold M diffeomorphic to $S\times\mathbb{R}$, where S is a closed, oriented surface of genus at least 2, which contains a non-empty, compact, geodesically convex subset, see [Thu80]. Such a manifold has a boundary at infinity, which is the union of two copies of S. Each of those two copies has a conformal structure, τ_+ and τ_-, induced by the hyperbolic metric on M. A celebrated theorem of Bers [Ber60, AB60] asserts that the map sending a quasifuchsian metric to (τ_+,τ_-) determines a parameterization of the space of quasifuchsian metrics on a M by the product of two copies of the Teichmüller space of S.

A quasifuchsian manifold M contains a smallest non-empty geodesically convex subset, called its convex core $C(M)$. The boundary of $C(M)$ is again the union of two copies of S, and each is a pleated surface in M, with a hyperbolic induced metric m_+, m_- and a measured bending lamination λ_+, λ_-. (There is a special, "Fuchsian" case, where $C(M)$ is a totally geodesic surface, the two pleated surfaces mentioned here are then the same, $m_+ = m_-$, and $\lambda_+ = \lambda_- = 0$.) It is known that any two hyperbolic metrics can be obtained in this way, this follows from [EM86] or from [Lab92], however it is not known whether any couple (m_+, m_-) can be uniquely obtained. Similarly, any two measured laminations λ_+, λ_- can be obtained in this manner if the weight of any leaf is less than π and if λ_+ and λ_- fill S [BO04], but it is not known whether uniqueness holds. Recall that λ_- and λ_+ fill S if there exists $\epsilon > 0$ such that, for any closed curve c in S, $i(\lambda_-, c) + i(\lambda_+, c) \geq \epsilon$.

Note that all the results mentioned here are actually known in the more general context of convex co-compact hyperbolic manifolds, i.e., interiors of compact manifolds with boundary, with a complete hyperbolic metric, containing a non-empty, compact, geodesically convex subset (or, even more generally for geometrically finite hyperbolic manifolds), the result concerning the measured bending lamination of the boundary can then be found in [Lec06]. We stick here to the quasifuchsian setting for simplicity.

Cone-manifolds. We consider here hyperbolic cone-manifolds of a special kind, which have cone singularities along curves (a more general notion is defined in [Thu80], allowing for singularities along graphs). Let $\theta \in (0, \pi)$, we call H^3_θ the hyperbolic manifold with cone singularities obtained by gluing isometrically the two faces of a hyperbolic wedge of angle θ (the closed domain in H^3 between two half-planes having the same boundary line). There is a unique such gluing which is the identity on the "axis" of the wedge. We will be using here the following (restrictive) definition.

Definition 1.1. *A hyperbolic cone-manifold is a manifold along with a metric for which each point has a neighborhood modeled on H^3_θ for some $\theta \in (0, \pi)$.*

Let M be a hyperbolic cone-manifold, it has two kind of points. Those which have a neighborhood isometric to a neighborhood of a point of some H^3_θ outside the cone singularity are called regular points, while the others are called singular points or cone points. The set of regular points will be denoted by M_r, and the set of singular points by M_s. By definition, M_s is a union of curves, if M is complete then those curves can be either closed curves or infinite lines. To each of those curves is associated an angle $\theta \in (0, \pi)$ – such that all points have a neighborhood isometric to a neighborhood of the cone singularity in H^3_θ – which is called its cone angle or simply its angle.

Recall the usual notion of convexity, which differs from other possible notions (e.g. the local convexity of the boundary of a domain).

Definition 1.2. *Let M be a hyperbolic cone-manifold. A subset $K \subset M$ is* **geodesically convex** *if any geodesic segment in M with endpoints in K is contained in K.*

Quasifuchsian manifolds with particles. Quasifuchsian manifolds with particles are defined in the same way as non-singular quasifuchsian manifold.

Definition 1.3. *A* **quasifuchsian manifold with particles** *is a complete hyperbolic cone-manifold M isometric to the product $S \times \mathbb{R}$, where S is a closed, orientable surface of genus at least 2, endowed with a complete hyperbolic metric with cone singularities on the lines $\{x_i\} \times \mathbb{R}$, for $x_1, \cdots, x_{n_0}$ distinct points in S, which contains a non-empty, compact, geodesically convex subset.*

Quasifuchsian manifolds with particles are always considered here up to isotopies.

Convex co-compact manifolds with particles. The previous definition can be extended to a definition of convex co-compact manifolds with particles.

Definition 1.4. *A* **convex co-compact hyperbolic manifold with particles** *is a complete hyperbolic cone-manifold M such that:*

- *M is homeomorphic to the interior of a compact manifold with boundary N, with non-trivial fundamental group,*
- *the singular locus corresponds under the homeomorphism to a disjoint union of curves in N with endpoints on ∂N,*
- *the angle at each singular curve is less than π,*
- *M contains a non-empty compact subset which is convex.*

A further extension to geometrically finite manifolds with particles is possible, we leave the details to the interested reader. We consider here only quasifuchsian mainfolds (with particles) although some of the intermediate statements can be extended to convex co-compact manifolds with particles. There is also some hope to extend the main results to this more general setting, however some technical hurdles have to be overcome before this can be achieved.

Note that there is another possible notion of quasifuchsian manifolds with cone singularities: those which are singular along closed curves, as studied in particular by Bromberg [Bro04b, Bro04a]. Although there are similarities between those two kinds cone-manifolds (in particular concerning their rigidity), the questions considered here are quite different from those usually associated to those considered for quasifuchsian cone-manifolds with singularities along closed curves (drilling of geodesics, etc).

1.2 The conformal structure at infinity.

Conformal structures and hyperbolic metrics on surfaces. Let's fix some notations.

Definition 1.5. *Let S be a closed surface, let $x_1, \cdots, x_{n_0} \in S$ be distinct points, and let $\theta_1, \cdots, \theta_{n_0} \in (0, \pi)$. Suppose that*

$$2\pi\chi(S) - \sum_{i=1}^{n}(2\pi - \theta_i) < 0 \ .$$

We then call:

- *$\mathcal{T}_{S,x}$ the space of conformal structures on S, considered up to isotopies of S fixing the x_i,*
- *$\mathcal{H}_{S,x,\theta}$ the space of hyperbolic metrics on S, with cone singularities at the x_i where the angle is θ_i, considered up to isotopies fixing the x_i.*

There is a one-to-one map between $\mathcal{T}_{S,x}$ and $\mathcal{H}_{S,x,\theta}$, because any conformal structure contains a unique hyperbolic metric with cone singularities at the x_i of prescribed angle (see [Tro91]). We keep distinct notations for clarity.

The conformal structure at infinity. Non-singular quasifuchsian manifolds have a natural conformal structure at infinity, which can be defined by considering the action of their fundamental group on their discontinuity domain, see [Thu80]. This definition cannot be used directly for quasifuchsian manifolds with particles, however it is still possible to define a conformal structure at infinity, see [MS06].

Therefore, to each quasifuchsian metric $g \in \mathcal{QF}_{\theta_1,\cdots,\theta_{n_0}}$ are associated two points $\tau_+, \tau_- \in \mathcal{T}_{S,n_0}$ corresponding to the conformal structures – marked by the endpoints of the "particles" – on the upper, resp. lower, connected component of the boundary at infinity.

A compactness lemma for the conformal structure at infinity. We consider again a closed surface S along with n distinct points ($n \geq 1$) $x_1, \cdots, x_{n_0} \in S$ and angles $\theta_1, \cdots, \theta_{n_0} \in (0, \pi)$ so that

$$2\pi\xi(S) - \sum_{i=1}^{n}(2\pi - \theta_i) < 0 \ .$$

Proposition 1.6. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of quasifuchsian metrics on $S\times\mathbb{R}$, with particles (cone singularities) on the lines $\{x_i\}\times\mathbb{R}$, of angle equal to θ_i. Suppose that the conformal structures at infinity, $\tau_{-,n},\tau_{+,n}\in\mathcal{T}_{S,n_0}$, converge to conformal structures $\tau_{-,\infty},\tau_{+,\infty}$. Then $(g_n)_{n\in\mathbb{N}}$ has a subsequence converging to a quasifuchsian metric with particles.*

The proof is contained in Section 6.2, it is based on the compactness results described below (in Section 3) relative to the induced metric and bending lamination on the boundary of the convex core.

A Bers-type theorem with particles. Using the previous proposition, along with the main result of [MS06], leads to an extension to quasifuchsian manifolds with particles of a classical result of Bers [Ber60] on "double uniformization".

Theorem 1.7. *The map from $\mathcal{QF}_{S,x,\theta}$ to $\mathcal{T}_{S,x}\times\mathcal{T}_{S,x}$ sending a quasifuchsian hyperbolic metric to the conformal structures at $+\infty$ and at $-\infty$ is a homeomorphism.*

1.3 The geometry of the convex core

Measured laminations We refer the reader to e.g. [CB88, PH92, Ota96] for the definition and main properties of measured laminations on closed (non-singular) surfaces. There are two possible definitions. One is geometric, in terms of measured geodesic laminations on hyperbolic surfaces, with a transverse measure, while the other definition is topological, and can involve the boundary at infinity of the surface. The two definitions are equivalent, basically because, in a closed (or finite volume) hyperbolic surface, any closed curve can be realized uniquely as a closed geodesic.

A well-known fact concerning hyperbolic surfaces with cone singularities is that, as long as the cone angles are less than π, it remains true that any homotopy class of closed curves contains a unique geodesic (see e.g. [DP07]). A fairly direct consequence is that, as for closed surfaces, any topological measured lamination (in the complement of the cone singularities) can be uniquely realized as a measured geodesic lamination.

Proposition 1.8. *Let Σ be a hyperbolic surface with cone singularities, where the angle is less than π. Let λ be a (topological) lamination on Σ. Then λ can be realized uniquely as a geodesic lamination.*

The space of measured geodesic laminations on a hyperbolic surface with cone singularities of angles less than π therefore does not depend on the cone angles.

Definition 1.9. *We call $\mathcal{ML}_{S,n_0}$ the space of measured lamination on S with n marked points.*

Thus, for any hyperbolic metric m on S with n cone singularities of angle less than π, $\mathcal{ML}_{S,n_0}$ can be canonically identified with the space of measured geodesic laminations on (S,m).

The convex core. The following basic proposition can be found in the appendix of [MS06].

Proposition 1.10. *Let M be a convex co-compact hyperbolic manifold with particles, and let K and K' be two non-empty geodesically convex subsets. Then $K\cap K'$ is a non-empty geodesically convex subset.*

It leads to a natural definition.

Definition 1.11. *Let M be a quasifuchsian manifold with particles. Its* **convex core** *$C(M)$ is the smallest non-empty geodesically convex subset contained in it.*

By construction, $C(M)$ is a "minimal" convex subset of M and it follows from general arguments (see [Thu80]) that its boundary is, outside the singular curves, a pleated surface (a locally convex, ruled surface). It turns out that, under the condition that the cone angles are less than π, the boundary of $C(M)$ intersects the cone singularities orthogonally, and is even totally geodesic in the neighborhood of each such intersection, see [MS06].

Therefore, given a quasifuchsian metric with particles $g\in\mathcal{QF}_{S,\theta}$, the induced metrics on the upper and lower boundary components of $C(M)$ (which might coincide in special cases) are two hyperbolic metrics $m_+,m_-\in$

$\mathcal{H}_{S,x,\theta}$. Moreover, those two boundary components are pleated along measured bending laminations $l_+, l_- \in \mathcal{ML}_{S,n_0}$.

The definition of the convex core extends to convex co-compact manifolds with particles. The boundary of this convex core is then still a pleated surface, orthogonal to the particles. So its induced metric is hyperbolic with cone singularities, and it is pleated along a measured lamination.

1.4 Prescribing the bending lamination

Results in the non-singular case. For non-singular convex co-compact hyperbolic manifolds an existence and uniqueness theorem for metrics with a given rational measured bending lamination was proved by Bonahon and Otal [BO04]. (Recall that a lamination is rational if its support is a disjoint union of closed curves.) When the lamination is not rational, an existence result was proved in [BO04] for manifolds with incompressible boundary, it was extended in [Lec06] to manifolds with compressible boundary.

Rational laminations with particles. As for quasifuchsian manifolds (without particles), it is possible to give an existence and uniqueness statement concerning the bending lamination on the boundary of the convex core, but only for rational lamination.

Theorem 1.12. *Let S be a closed surface, let $x_1, \cdots, x_{n_0} \in S$ be distinct points, and let $\theta_1, \cdots, \theta_{n_0}$. Let $\lambda_-, \lambda_+ \in \mathcal{ML}_{S,x}$ be measured laminations, each with support a disjoint union of closed curves. Suppose that:*

- *λ_- and λ_+ fill S,*
- *each closed curve in the support of λ_- (resp. λ_+) has weight less than π.*

Then there exists a metric $g \in \mathcal{QF}_{S,x,\theta}$ such that the measured bending lamination on the upper (resp. lower) boundary component of the convex core of $(S \times \mathbb{R}, g)$ is λ_+ (resp. λ_-). Moreover g is unique up to isotopies.

The proof, which is given in Section 4, is based on the rigidity theorem of Hodgson and Kerckhoff [HK98] for closed hyperbolic manifolds with cone singularities. We prove in Lemma 4.3 that the hypothesis are necessary conditions.

General laminations. When considering laminations which are not necessarily rational, we obtain only a weaker result. First, because we can only claim existence, but not uniqueness (this remains an open problem even in the non-singular case, see [BO04, Lec06]). Second, because we can only treat quasifuchsian manifolds with particles, and not more generally convex co-compact manifolds.

Theorem 1.13. *Let S be a closed surface, let $x_1, \cdots, x_{n_0} \in S$ be distinct points, and let $\theta_1, \cdots, \theta_{n_0}$. Let $\lambda_-, \lambda_+ \in \mathcal{ML}_{S,x}$. Suppose that:*

- *λ_- and λ_+ fill S,*
- *each closed curve in the support of λ_- (resp. λ_+) has weight less than π.*

Then there exists a metric $g \in \mathcal{QF}_{S,x,\theta}$ such that the measured pleating lamination on the upper (resp. lower) boundary component of the convex core of $(S \times \mathbb{R}, g)$ is λ_+ (resp. λ_-).

The conditions in this theorem are easily seen to be necessary, see Lemma 4.3.

1.5 The induced metric on the boundary of the convex core.

The Bers-type result on the conformal metric at infinity can be used to obtain an existence result concerning the prescription of the induced metric on the boundary of the convex core.

Theorem 1.14. *Let $m_-, m_+ \in \mathcal{H}_{S,n_0,\theta}$, where $\theta = (\theta_1, \cdots, \theta_{n_0}) \in (0, \pi)^n$. There exists a quasifuchsian metric with particles on $S \times \mathbb{R}$, with particles of angle θ_i at the lines $\{x_i\} \times \mathbb{R}$, for which the induced metric on the boundary of the two connected components of the convex core are m_- and m_+.*

In the smooth case – i.e. for quasifuchsian hyperbolic manifolds without conical singularities – the corresponding result is well-known, it follows either from results of Labourie [Lab92] or from a partial answer, first given by Epstein and Marden [EM86], to a conjecture of Sullivan.

As for the conformal structure at infinity, it might be possible to extend this statement to cover convex co-compact (resp. geometrically finite) manifolds with particles. The uniqueness remains elusive, as in the non-singular case.

1.6 Applications

Quasifuchsian manifolds can be used as tools in Teichmüller theory. By extension, the quasifuchsian manifolds with particles considered here can be used as tools for the study of the Teichmüller space of hyperbolic metrics with cone singularities (of angle less than π) on a surface.

One such application is through the renormalized volume of those quasifuchsian manifolds with particles, as considered in [KS08a, KS09]. In the non-singular case this renormalized volume is equal to the Liouville functional (see [TZ87, TT03, TZ03]), it is a Kähler potential on $\mathcal{H}_{S,x,\theta}$. Other applications of closely related tools, in the non-singular context, for the global geometry of the Weil-Petersson metric on Teichmüller space, can be found in [McM00]. Yet other applications, to some properties of the grafting map, are considered in [KS08b], and the manifolds with particles considered here should allow for an extension to the grafting map on $\mathcal{H}_{S,x,\theta}$.

1.7 Outline of the proofs

We now turn to a description of the main technical points of the proofs.

Measured bending laminations. Theorem 1.12 is proved by an argument strongly influenced by the proof given by Bonahon and Otal [BO04] for non-singular convex co-compact manifolds. Thanks to a doubling trick, the infinitesimal rigidity of the convex cores of convex co-compact manifolds with particles, with respect to the (rational) measured bending lamination, is reduced to an important infinitesimal rigidity result proved for hyperbolic cone-manifolds by Hodgson and Kerckhoff [HK98]. A deformation argument then provides the proof of the theorem.

The existence result for general laminations on quasifuchsian manifolds with particles (Theorem 1.13) can then be obtained by an approximation argument, as in the non-singular case in [BO04, Lec06]. The key step of the proof is a compactness statement, showing that if the measured bending laminations converge to a limit having good properties, then the quasifuchsian metrics converge after extracting a subsequence. However the arguments developed in [BO04, Lec06] cannot be used in the context of quasifuchsian manifolds with particles, because they rely heavily on the representation of the fundamental group. Different arguments are therefore used here, which are more differential-geometric in nature.

Those arguments are sometimes technically involved because of the added difficulties induced by the particles. However, after stripping the proof of the elements which are needed only because of the particles (for instance the multiple cover argument used in Section 3.4 to find simplicial surfaces with given boundary in the convex core), the compactness proof given here is simpler than the one in [BO04, Lec06].

Prescribing the induced metric on the boundary of the convex core. We give in Section 6 a rather elementary proof of Theorem 1.14, which has two parts. Call t_- (resp. t_+) the hyperbolic metric in the conformal class τ_- (resp. τ_+) with cone angles θ_i at x_i. The first part is an upper bound on the length of the curves in the hyperbolic metric at infinity $t_\pm$, following [Bri98, BC03].

Now recall that, by Thurston's Earthquake Theorem [Ker83, Thu86a], there exists a unique right earthquake sending a given hyperbolic metric to another one. This extends to hyperbolic metrics with cone singularities of angle less than π, see [BS06]. In particular there is a unique measured lamination ν_+ such that the right earthquake along ν_+, applied to m_+, yields the hyperbolic metric t_+. The second part of our proof is a bound on the length of ν_+ for m_+ (see Proposition 5.2).

This is then used in Section 6 to prove Theorems 1.7 and 1.14.

Quasi-conformal estimates. There is another possible way to prove Theorem 1.14, closer to the argument used in the non-singular case (as seen in [EM86, Bri98, BC03]). It uses a bound on the quasi-conformal factor between the conformal structure at infinity τ and the conformal class of the induced metric m on the boundary of the convex core, both understood as elements of $\mathcal{T}_{S,n_0}$.

Proposition 1.15. *There exists a constant $C > 0$ (depending only on the topology of M) such that τ is C-quasiconformal to m.*

This proposition is not formally necessary to obtain the main results presented here, it can be found in Appendix A.

As mentioned above, the proof of Theorem 1.14 through Proposition 1.15 would be much closer to the proof(s) known in the non-singular case. It can be pointed out that the proof given in Section 6 is quite parallel, but in the context of Teichmüller theory understood as the study of hyperbolic rather than complex surfaces. From this viewpoint, Proposition 5.2 is a direct analog of Proposition 1.15, with quasiconformal deformations replaced by earthquakes.

What follows. Section 2 presents the definition of the convex core of a convex co-compact manifold with particles, and some of its simple properties, extending well-known properties with no cone singularity. In section 3 we state and prove a key compactness statement with respect to the measured bending lamination on the boundary of the convex core. Section 4 contains the proof of Theorem 1.12, using a local rigidity statement of Hodgson and Kerckhoff [HK98] and the compactness lemma of Section 3. Section 5 contains the proof of Theorem 1.13, and Section 6 contains the proof of Theorem 1.7 and of Theorem 1.14. Section 7 contains some remarks on the analogy with corresponding problems in anti-de Sitter geometry and on applications to the Weil-Petersson metric of the Teichmüller space of hyperbolic metrics with cone singularities of prescribed angles on a closed surface (see [KS08a, KS09, KS08b]). Finally, Appendix A contains the proof of Proposition 1.15, based on the estimates on the length of the earthquake lamination obtained in Section 5.

2 The geometry of the convex core

This section contains some basic statements necessary to understand the geometry of convex co-compact manifolds with particles, concerning in particular the convex core and its boundary. We consider here such a convex co-compact manifold with particles, M, and denote by M_r its regular part and by M_s its singular part (the union of the singular lines).

2.1 Surfaces orthogonal to the singular locus

We define here a natural notion of pleated surface orthogonal to the singular locus in M. The first step is to define the notion of totally geodesic plane orthogonal to a cone singularity in a hyperbolic cone-manifold. The first condition is that the surface is totally geodesic outside its intersections with the particles. The second condition is local, in the neighborhood of the intersections with the particles; there, the surface should correspond to the image in H^3_θ of the restriction to the wedge (used to define H^3_θ) of a plane orthogonal to the axis of the wedge.

Definition 2.1. *Let Σ be a pleated surface in M_r, and let Σ' be its closure as a subset of M; suppose that $\Sigma' \setminus \Sigma \subset M_s$. We say that Σ' is orthogonal to the singular locus if any $x \in \Sigma' \setminus \Sigma$ has a neighborhood in Σ' which is a totally geodesic surface orthogonal to the singular locus.*

This definition can be extended to encompass more general surfaces, i.e., surfaces which are neither pleated nor totally geodesic in the neighborhood of the singular locus. In this more general case the definition can be given in terms of the convergence of the unit normal vector to a vector “tangent” to the singular locus at its intersection with the surface. This will however not be needed here.

2.2 The convex core of a manifold with particles

Among the defining properties of a quasifuchsian cone-manifold M is the fact that it contains a compact subset K which is convex in the (strong) sense that any geodesic segment in M with endpoints in K is contained in K. We have already seen that it is possible to define the *convex core* of M as the smallest compact subset of M which is convex, denoted by $C(M)$.

Theorem 2.2. *Suppose that $C(M)$ is not a totally geodesic surface. Then its boundary is the disjoint union of surfaces which are orthogonal to the singular locus. Each connected component of the singular locus of M intersects $C(M)$ along a segment.*

The proof is a consequence of two lemmas, both stated under the hypothesis of the theorem. The second lemma in particular gives more precise informations on the geometry of the convex core.

Lemma 2.3. *The boundary of $C(M)$ is a surface orthogonal to the singular locus.*

The proof can be found in the appendix of [MS06].

Let $x \in M$, we denote by L_x the **link** of M at x, that is, the space of geodesic rays starting from x (parametrized at speed 1), with its natural angle distance. When x is a regular point of M, L_x is isometric to the 2-dimensional sphere S^2, with its round metric. When x is contained in a singular line of angle θ, L_x can be described as the metric completion of the quotient by a rotation of angle θ of the universal cover of the complement of two antipodal points in S^2.

Definition 2.4. *Let $K \subset M$ be convex, and let $x \in M$. The* **link** *of K at x is the set of vectors $v \in L_x$ such that there is a (small) geodesic ray starting from x in the direction of v which is contained in K. It is denoted by $L_x(K)$.*

Clearly $L_x(K) = \emptyset$ when x is not contained in K, while $L_x(K) = L_x$ when x is contained in the interior of K.

To go further, we define the oriented normal bundle of $\partial C(M)$, denoted by $N_r^1\partial C(M)$, as the set of $(x, n) \in TM$ such that $x \in C(M)$ is not in the singular locus of M and n is a unit vector such that its orthogonal is a support plane of $C(M)$ at x, and n is oriented towards the exterior of $C(M)$.

Let $x \in M$ be a non-singular point, let $v \in T_xM$ and let $t \in \mathbb{R}_+$. For t small enough, it is possible to define the image of (x, tv) by the exponential map, it is the point $\exp(x, tv) := g(t)$, where g is the geodesic, parametrized at constant speed, such that $g(0)x$ and $g'(0) = v$. As t grows, $\exp(x, tv)$ remains well-defined until g intersects the singular set of M.

Lemma 2.5. *The exponential map is a homeomorphism from $N_r^1\partial C(M) \times (0, \infty)$ to the complement of $C(M)$ in M, and its restriction to the complement of the points of the form (x, v, t), for $x \in M_s$ and v a singular point of L_x, is a diffeomorphism to complement of $C(M)$ in M_r. The map:*

$$\begin{array}{llll} \exp_\infty : & N_r^1\partial C(M) & \to & \partial_\infty M \\ & (x, v) & \mapsto & \lim_{t\to\infty} \exp(x, tv) \end{array}$$

is a homeomorphism from $N_r^1\partial C(M)$ to the complement in $\partial_\infty M$ of the endpoints of the singular curves in M.

The proof can also be found in the appendix of [MS06].

The proof of Theorem 2.2 clearly follows from Lemma 2.3 and Lemma 2.5, since Lemma 2.5 shows that the cone singularities cannot re-enter the convex core after exiting it.

2.3 The geometry of the boundary

By construction, $C(M)$ is a minimal convex set in M, and it follows as in the non-singular case (see [Thu80]) that its boundary is a “pleated surface” except at its intersections with the singular curves.

Lemma 2.6. *The surface $\partial C(M)$ has an induced metric which is hyperbolic (i.e. it has constant curvature -1) with conical singularities at the intersections of $\partial C(M)$ with the singular curves of M, where the total angle is the same as the total angle around the corresponding singular curve. It is "pleated" along a measured lamination λ in the complement of the singular points. Moreover the distance between the support of λ and the intersection of the singular set of M with $\partial C(M)$ is strictly positive.*

Proof. Since $C(M)$ is a minimal convex subset, its boundary is locally convex and ruled, therefore developable (see [Spi75] for the Euclidean analog, or [Thu80]) so that its induced metric is hyperbolic. The fact that its intersection points are conical singularities, with a total angle which is the same as the total angle around the corresponding singularities, is a consequence of the fact that $\partial C(M)$ is orthogonal to the singularities.

Similarly, the fact that $\partial C(M)$ is pleated along a measured lamination is a direct consequence of the fact that it is ruled and locally convex, i.e. that each point in $\partial C(M)$ is in either a complete hyperbolic geodesics or a totally geodesic ideal triangle. The support of λ is a disjoint union of embedded maximal geodesics, and it is well-known (see e.g. [DP07]) that (under the hypothesis that the angles at the cone singularities are strictly less than π) embedded geodesics remain at positive distance from the singular locus. So the distance between the support of λ and the singular locus of $\partial C(M)$ is strictly positive. □

2.4 The distance between the singular curves

We state and prove here some elementary statements on the distance between singular points in the boundary of $C(M)$ and between singular curves in M. They will be useful at several points below.

Lemma 2.7. *Let $\theta \in (0, \pi)$. There exists $\epsilon > 0$ and $\rho > 0$, depending on θ, such that:*

1. *in a complete hyperbolic surface with cone singularities of angle less than θ (not homeomorphic to a sphere), two cone singularities are at distance at least ϵ,*
2. *if M is a convex co-compact hyperbolic manifold with particles of angle less than θ, then any two particles in M are at distance at least ϵ,*
3. *if D is a closed 2-dimensional geodesic disk of radius ϵ centered at a singular point x_0 of cone angle θ, and if $\Omega \subset D$ is a convex subset whose closure intersects the boundary of D, then Ω contains all points of D at distance at most ρ from x_0.*

In particular, it follows from point (1) that no embedded geodesic in D can come within distance less than ρ from the cone singularity.

Proof. The first point is well-known, see e.g. [DP07]. The interested reader can construct an elementary proof based on Dirichlet domains, as in 3-dimensional manifolds in the proof of the second point, below.

For the second point we use an argument which is well-known for closed hyperbolic cone-manifolds with cone angles less than π, see e.g. [BP01]. Choose a point x_0 in the regular part of M, and let D_{x_0} be the set of points $x \in M_r$ such that there exists a unique minimizing geodesic segment from x_0 to x in M_r. It follows from its definition that D_{x_0} is contractible, and it is not too difficult to show that it is isometric to a convex infinite polyhedron in H^3, with the cone singularities of M corresponding to some of the edges. The fact that the cone angles are less than θ then implies that the dihedral angles of this polyhedron are less than $\theta/2$ (which is less than $\pi/2$). The result then follows from the fact that, in polyhedron of angles less than $\theta/2$, the distance between two non-adjacent edges is bounded from below by a constant depending only on θ.

For the third point let x_1 be a point in $\partial D \cap \Omega$, and let γ be the minimizing geodesic segment from x_0 to x_1. Since D contains no other singular point by the first point, so the complement of γ in D is isometric to an angular sector in the disk of radius ϵ in H^2. This angular sector has three vertices, one corresponding to x_0 and the other two corresponding to x_1. Since $\theta < \pi$, it is convex at the vertex corresponding to x_0. Let s be the geodesic segment joining the two vertices corresponding to x_1. Then Ω, being convex, contains the projection in D of the triangle bounded by s and by the two geodesic segments in the boundary of D joining x_0 to the two vertices projecting to x_1. This proves the statement, with ρ equal to the distance between x_0 and s. □

We now call $\epsilon_0 > 0$ the number ϵ associated by this lemma to the maximum of the θ_i, and ρ_0 the corresponding value of ρ.

3 Compactness statements

3.1 Main statement.

The main goal of this section is to prove the following compactness lemma.

Lemma 3.1. *Let $N = S \times I$, let $p_1, \cdots, p_{n_0}$ be distincts points on S, and let $\kappa_i = \{p_i\} \times I, 1 \leq i \leq n_0$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of quasifuchsian metrics on N with a particle of angles $\theta_n^i \in (0, \pi)$ along κ_i, $1 \leq i \leq n_0$. Let λ_n be the measured bending laminations on the boundary of the convex core of N (considered as a measured lamination on ∂N minus the endpoints of the κ_i). Suppose that $\theta_n^i \to \theta^i \in (0, \pi)$ for all $i \in \{1, \cdots, n_0\}$, and that $\lambda_n \to \lambda_\infty$, where λ_∞ satisfies the hypothesis of Theorem 1.13. Then, after taking a subsequence, (g_n) converges to a quasifuchsian metric on N, with cone singularities of angle θ^i at the κ_i, such that the bending lamination of the boundary of the convex core is λ_∞.*

Although this lemma is a generalisation of the "Lemme de fermeture" of [BO04], the proof is very different. The point is that the 2 main ingredients of the proof in Bonahon-Otal's paper are Culler-Morgan-Shalen compactification of the character variety by actions on $\mathbb{R}$-trees and the covering Theorem of Canary. Since both these result hardly extend to manifolds with particles we had to use different arguments. Since our proof also works without particles, we get a new proof of the main result of [BO04].

3.2 A finite cover argument

We work under the assumption that the cone angle around each singularity is less than π. Thanks to that assumption, we can guarantee that the singularities are never too close to each other, see Lemma 2.7, and that the boundary of the convex core is well defined and is orthogonal to the singularities. On the other hand, cone singularities with cone angles less than π can be viewed has singularities with concentrated positive curvature. But some of the results we will use are easier to prove when the curvature is bounded above. To overcome this difficulty, we will use a branched cover for which the cone angles are all at least 2π.

For each singularity κ_i, we choose an integer n_i such that $\frac{2\pi}{n_i}$ is less than the angle θ^i of the cone singularity along κ_i in the limit as $n \to \infty$. The surface S with cone angle $\frac{2\pi}{n_i}$ at the point p_i is a hyperbolic orbifold. As such it has a manifold cover $h : \bar{S} \to S$ which is a branched cover so that the lifts of the point p_i have a branching index equal to n_i. The branched cover $h : \bar{S} \to S$ extends naturally to a branched cover $h : \bar{S} \times \mathbb{R} \to S \times \mathbb{R}$. For a fixed n, if we pull back the metric g_n using the map h, we get a hyperbolic metric with cone singularities on $\bar{N} = \bar{S} \times \mathbb{R}$ for which the covering transformations are isometries. Let $\bar{M} = (\bar{S} \times \mathbb{R}, \bar{g}_n)$ be the manifold with cone singularities thus obtained, and let $\bar{C}(g)$ be its convex core. This set $\bar{C}(g)$ cover $C(g)$. By the choice of the n_i, the cone angle around each singularity of $\bar{M}$ is at least 2π for n large enough. Such a singularity can be viewed as a set of concentrated negative curvature. More precisely $\bar{M}$ can be approximated by Riemannian manifolds with curvature bounded above (in the bilipschitz topology). It follows that $(\bar{M}, \bar{g}_n)$ has some properties of a negatively curved manifold, for instance the uniqueness of the geodesic segment joining two given points, or the Margulis Lemma.

3.3 Pleated annuli

A technical device which will be useful below is a simplicial surface bounded by two given curves. Let us construct such an object. Let $\bar{d}, \bar{d}'$ be simple closed geodesics, respectively on the upper and on the lower boundary component of $\bar{C}(g)$, which are homotopic in the complement of the singular locus of $\bar{M}$. We fix n and denote $g = g_n$.

Lemma 3.2. *There exists an immersed annulus $\bar{A}$ bounded by $\bar{d} \cup \bar{d}'$ such that the metric induced on $\bar{A}$ is a Riemannian metric with curvature ≤ -1 with cone singularities with angles at least 2π and the area of $\bar{A}$ is at most $\ell(\bar{d}) + \ell(\bar{d}')$.*

Proof. Since $\bar{d}$ and $\bar{d}'$ are homotopic simple closed curves, by the annulus Theorem (see [Wal67]), there is an embedded annulus $\bar{A} \subset \bar{C}(g)$ with $\partial\bar{A} = \bar{d} \cup \bar{d}'$. If the bending laminination of $\bar{C}(g)$ intersects $\bar{d}$ and $\bar{d}'$ finitely

many times, then $\bar{d}$ and $\bar{d}'$ are piecewise geodesics. If not, we approximate them by piecewise geodesic curves and work on the approximates. Consider a triangulation T of $\bar{A}$ whose vertices are all contained in $\bar{d} \cup \bar{d}'$ and such that any vertex of $\bar{d}$ and $\bar{d}'$ (when considered as piecewise geodesics) is a vertex of T. As we have said before, in $\bar{M}$, there is a unique geodesic segment joining 2 given points. It follows that we can change $\bar{A}$ by a homotopy so that each edge of T is a geodesic segment in $\bar{C}(g)$. Next, for each triangle T_k of T, we choose a vertex v and we substitute T by the geodesic cone from v to the edge not containing v. This geodesic cone is the union of the geodesic segments joining v to the edge e_v of T_k that does not contain v. Again the existence of this cone follows from the uniqueness of geodesic paths. From now on we denote this cone by T_k. By construction, it is a locally ruled surface and as such has negative curvature :

Claim 3.3. *The surface T_k is an union of polygons with curvature less than -1. Furthermore, the sum of the angles of the polygons meeting at an interior vertex is at least 2π.*

Proof. The surface T_k meets the singular locus $\bar{M}_s$ of $\bar{M}$ along segments and at points. For each component κ of $\bar{M}_s \cap T_k$ we consider the two extremal segments joining v to e_v and intersecting κ. Doing this for each component of $\bar{M}_s \cap T_k$, we get a family of segments which are geodesic for the metric of $(\bar{M}, \bar{g})$ and hence for the induced metric on T_k. We add the components of $\bar{M}_s \cap T_k$ which are segment to this family and get a new family of geodesic segments. The closure of each complementary region is a polygon, i.e. a disc with piecewise geodesic boundary (see Figure 3.3). By construction each such polygon is a locally ruled surface. The curvature of a locally ruled surface is at most the sectional curvature of the ambient space, which is -1 here. Thus we have proved the first sentence of this claim.

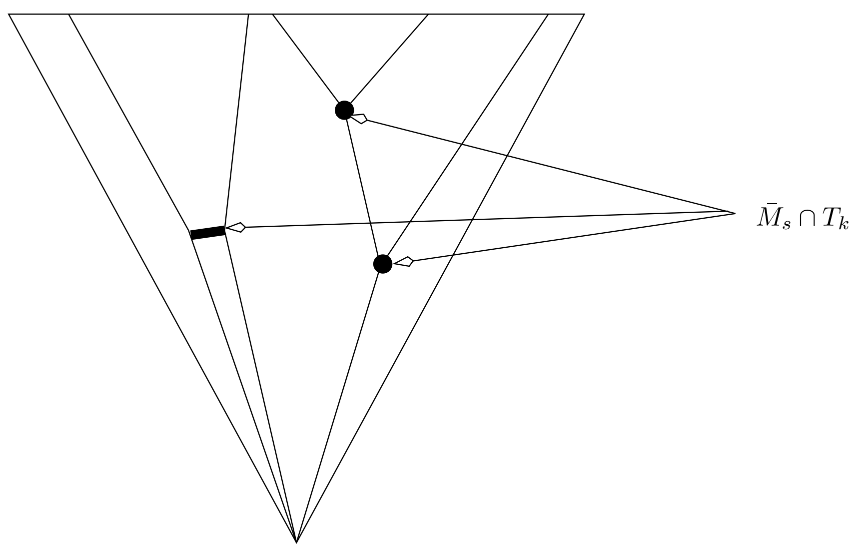

Figure 1: Decomposition of T_k into hyperbolic polygons

Since no two components of the singular locus intersect each other, any vertex of any polygons lies in at most one component of the singular locus. Furthermore, by construction, given an interior vertex v of this decomposition into polygons, there is a geodesic segment (for the metric of $(\bar{M}, \bar{g})$) which passes through v. On each side of this segment, the sum of the angles of the polygons has to be at least π. Thus, we can conclude that the sum of the angles of the polygons around v is at least 2π. $\square$

It remains to prove that $\text{Area}(\bar{A}) \leq \ell(\bar{d}) + \ell(\bar{d}')$. By construction, $\bar{A}$ is a union triangles T_k such that one edge of each T_k lies in $\bar{d} \cup \bar{d}'$ and by Claim 3.3, the induced metric on each such triangle is at most -1. Let T_k^h be a hyperbolic triangle (i.e. a geodesic triangle in $\mathbb{H}^2$) such that the length of the edges of T_k^h are the same as the length of the edges of T_k. Since the induced metric on T_k as curvature ≤ -1, we have $\text{Area}(T_k) \leq \text{Area}(T_k^h)$. On the other hand the area of a hyperbolic triangle is less than the length of any of its edges (see [Thu80,

Lemma 9.3.2]). It follows that Area(T_k) is less than the length of any of its edges, in particular it is less than the length of the edge of T_k lying in $\bar{d} \cup \bar{d}'$). Since this holds for all the triangles composing $\bar{A}$, we have Area($\bar{A}$) $\leq \ell(\bar{d}) + \ell(\bar{d}')$. ☐

It follows from Claim 3.3 and the Gauss-Bonnet Formula, that the area of $\bar{A}$ is at most the bending of $\partial\bar{A}$, namely Area($\bar{A}$) $\leq i(\partial\bar{A}, \bar{\lambda}_n) = i(\bar{d}, \bar{\lambda}_n) + i(\bar{d}', \bar{\lambda}_n)$.

3.4 Long geodesics in M.

In this section, we will show that, under the hypothesis of Lemma 3.1, the induced metrics on $\partial C(g_n)$ are bounded. In order to do that we will show that if some geodesic is long in the boundary of $C(g)$, then the boundary of some annulus is almost not bent. Since the condition on λ_∞ provides us with a uniform bound on the minimal bending of an annuli, this will provide us with a bound on the length of any given simple closed curve on $\partial C(g_n)$.

Let us denote by S and S' the 2 components of ∂M. Since $C(g_n)$ is homeomorphic to M, $\partial C(g_n)$ is homeomorphic to $S \sqcup S'$. Let m_n be the metric defined on $S \sqcup S'$ by the identification with $\partial C(g_n)$ endowed with the metric induce by the g_n-length of paths. This metric m_n is a hyperbolic metric with cone singularities.

We will make use of the branched cover $\bar{M}$ defined in Section 3.2. Notice that since the angles along the cone singularities converge to limits in $(0, \pi)$, the lifted metric $\bar{m}_n$ on $\bar{M}$ is a hyperbolic metric with cone singularities for which the angles around the singularities are all at least 2π.

The following lemma will allow us to say that the induced metric on $\partial C(g_n)$ is bounded.

Lemma 3.4. *Assume that (λ_n) converges to λ_∞ (without any hypothesis on λ_∞). Consider a simple closed curve $d \subset \partial C(g)$. Let $d_n \subset \partial C(g_n)$ be the closed m_n-geodesics freely homotopic to d (in $C(g_n)$) lying in one component, say S, of $\partial C(g)$. Assume that $l_{m_n}(d_n) \longrightarrow \infty$. Then either λ_∞ contains a leaf with a weight of at least π, or there is a sequence of essential annuli E_n such that $i(\lambda_n, \partial E_n) \longrightarrow 0$.*

Proof. Let S' be the other boundary component of $C(M)$, and let d_n' be the closed m_n-geodesic freely homotopic to d lying in S'. We denote by $\bar{C}(g_n)$ the lift of $C(g_n)$ to $(\bar{M}, \bar{g}_n)$. Let $\bar{d}_n$ and $\bar{d}_n'$ be homotopic preimage of d_n and d_n' respectively to $\bar{C}(g_n)$ under the covering projection $\bar{M} \to M$. We denote by $\bar{\lambda}_n \in \mathcal{ML}(\partial\bar{M})$ the preimage of $\bar{\lambda}_n$ under the covering projection $\bar{M} \to M$. This measured geodesic lamination $\bar{\lambda}_n$ is the bending measured lamination of $\bar{C}(g_n)$. Furthermore $\bar{\lambda}_n$ converge to the preimage $\bar{\lambda}_\infty$ of λ_∞.

First we will show that $\bar{d}_n \cup \bar{d}_n'$ has points which are close to each other in $\bar{M}$ but far in $\bar{d}_n \cup \bar{d}_n'$. This can happen for instance if d_n and d_n' are close to each other in $C(g_n)$ but not in $\partial C(g_n)$.

Claim 3.5. *There are 2 points $\bar{x}_n \in \bar{d}_n$ and $\bar{y}_n \in \bar{d}_n \cup \bar{d}_n'$ and a geodesic arc $\bar{k}_n \subset \bar{C}(g_n)$ joining $\bar{x}_n$ to $\bar{y}_n$ such that $\ell_{\bar{m}_n}(\bar{k}_n) \longrightarrow 0$ and that either $\bar{y}_n \in \bar{d}_n'$ or there is a $\bar{m}_n$-geodesic arc $\bar{\kappa}_n \in \bar{d}_n$ that is homotopic to $\bar{k}_n$ relative to its boundary $\{\bar{x}_n\} \cup \{\bar{y}_n\}$ and that satisfies $\ell_{\bar{m}_n}(\bar{\kappa}_n) \longrightarrow \infty$.*

Proof. By assumption, $(\bar{\lambda}_n)_{n\in\mathbb{N}}$ converges to $\bar{\lambda}_\infty$. It follows that $i(\bar{d}_n \cup \bar{d}_n', \bar{\lambda}_n)$ is bounded. Consider the annulus $\bar{A}_n$ with $\partial\bar{A}_n = \bar{d}_n \cup \bar{d}_n'$ that was constructed in Lemma 3.2. Since $\bar{M}$ is a ramified finite cover, $l_{\bar{m}_n}(\bar{d}_n) \longrightarrow \infty$. Hence, there is $\varepsilon_n \longrightarrow 0$ and a segment $\bar{s}_n \subset \bar{d}_n$ such that $l_{\bar{m}_n}(\bar{s}_n) \longrightarrow \infty$ and $i(\bar{s}_n, \bar{\lambda}_n) \leq \varepsilon_n$. Let $\bar{t}_n \subset \bar{C}(g_n)$ be the $\bar{g}_n$-geodesic segment homotopic to $\bar{s}_n$ relative to its endpoints. Since $\bar{s}_n$ is almost not bend, its length is very close to the length of $\bar{t}_n$ (see [Lec06, Lemme A2]). In particular, $l_{\bar{g}_n}(\bar{t}_n) \longrightarrow \infty$. Furthermore, for the same reason, any point in $\bar{s}_n$ is close to $\bar{t}_n$. Namely there is $\eta_n = \eta(\varepsilon_n) \longrightarrow 0$ such that for any point $\bar{z}_n \subset \bar{s}_n$, there is $\bar{x}_n \subset \bar{t}_n$ with $d_{\bar{g}_n}(\bar{x}_n, \bar{z}_n) \leq \eta_n$ (see [Lec06, Affirmation A3]).

The bending $i(\bar{\lambda}_n, \partial\bar{A}_n)$ of $\partial\bar{A}_n$ is bounded. It follows from the Gauss-Bonnet formula that the area of $\bar{A}_n$ is bounded. Now, in $\bar{d}_n$, we replace $\bar{s}_n$ by $\bar{t}_n$. By the previous paragraph, we can still consider the annulus $\bar{A}_n$ and its area is bounded. For any point in $\bar{t}_n$ that is at distance at least $\frac{1}{3}l_{\bar{m}_n}(\bar{t}_n)$ from $\partial\bar{\kappa}_n$, we consider in $\bar{A}_n$ an arc orthogonal to $\bar{t}_n$ that either hits $\partial\bar{A}_n$ or has length η_n. Let $\bar{Z}_n \subset \bar{A}_n$ be the union of those arcs that have length η_n and let $\bar{z}_n$ be the union of their starting points (i.e. their intersection with $\bar{t}_n$). The set $\bar{Z}_n$ is embedded and its area is the same as the area of a strip of length $\ell_{\bar{m}_n}(\bar{z}_n)$ and width η_n. Notice that since the singularities of $\bar{A}_n$ have cone angles at least 2π, the area of this strip is not smaller than the area of a hyperbolic

strip with the same length and width, i.e. it is at least $\ell_{\bar{m}_n}(\bar{z}_n)\sinh(\eta_n)$. Let K be an upper bound for the area of $\bar{A}_n$. Taking η_n such that $\sinh(\eta_n) > \frac{3K}{\ell_{\bar{m}_n}(\bar{t}_n)}$, we get

$$K \geq \text{Area}(\bar{Z}_n) \geq \ell_{\bar{m}_n}(\bar{z}_n)\sinh(\eta_n) > K\frac{3\ell_{\bar{m}_n}(\bar{z}_n)}{\ell_{\bar{m}_n}(\bar{t}_n)} .$$

Hence $\ell_{\bar{m}_n}(\bar{z}_n) < \frac{1}{3}\ell_{\bar{m}_n}(\bar{t}_n)$. It follows that there exists an arc with length less than η_n orthogonal to $\bar{t}_n$ which hits $\partial\bar{A}_n$.

The endpoints of such an arc are 2 points $\bar{x}'_n \subset \bar{t}_n$ and $\bar{y}_n \subset \partial\bar{A}_n$ satisfying $d_{\bar{g}_n}(\bar{x}'_n, \bar{y}_n) \longrightarrow 0$. As we have seen in the previous paragraph there is a point $\bar{x}_n \subset \bar{k}_n$ very close to $\bar{x}'_n$. It follows that $\bar{x}_n \subset \bar{d}_n$ and $\bar{y}_n \subset \bar{d}_n \cup \bar{d}'_n$ are joined in $\bar{C}(g_n)$ by an arc $\bar{k}_n$ satisfying $\ell_{g_n}(\bar{k}_n) \longrightarrow 0$.

If $\bar{y}_n \in \bar{d}'_n$ then we are done. Otherwise $\bar{x}_n$ and $\bar{y}_n$ both lie in $\bar{d}_n$. By construction $\bar{k}_n$ lies in an annulus connecting $\bar{d}_n$ to $\bar{d}'_n$. It follows that there is a $\bar{m}_n$-geodesic arc $\bar{\kappa}_n \subset \bar{d}_n$ that is homotopic to $\bar{k}_n$ relative to $\{\bar{x}_n\} \cup \{\bar{y}_n\}$. It follows from the construction of $\bar{x}_n$ and $\bar{y}_n$ that $\bar{\kappa}_n \not\subset \bar{s}_n$. Since $\bar{x}_n$ it at distance at least $\frac{l_{\bar{m}_n}(\bar{t}_n)}{3} \longrightarrow \infty$ from the points in $\partial\bar{s}_n$, we have

$$\ell_{\bar{m}_n}(\bar{\kappa}_n) \geq \frac{l_{\bar{m}_n}(\bar{t}_n)}{3} \longrightarrow \infty .$$

□

Extract a subsequence such that either $\bar{y}_n \in \bar{d}_n$ for any n or $\bar{y}_n \in \bar{d}'_n$ for any n. We will show below that if $\bar{y}_n$ lies in $\bar{d}_n$ then λ_∞ has a leaf with a weight equal to π and that if $\bar{y}_n$ lies in $\bar{d}'_n$ then there is a sequence of essential annuli $E_n \subset M$ such that $i(\lambda_n, \partial E_n) \longrightarrow 0$.

Next we are going to show that an $\bar{m}_n$-geodesic loop based at $\bar{x}_n$ is almost not bent. Consider a $\bar{m}_n$-geodesic loop $\bar{l}_n$ based at $\bar{x}_n \subset \partial\bar{C}(g_n)$. We use the points $\bar{x}_n, \bar{y}_n \subset \partial\bar{C}(g_n)$ and arcs $\bar{k}_n$ and $\bar{\kappa}_n$ constructed in Claim 3.5. Let $\tilde{M}_n$ be the universal cover of $(\bar{M}, \bar{g}_n)$, it is a simply connected hyperbolic 3-manifold with cone singularities. Let $\tilde{C}(g_n)$ be the lift of $\bar{C}(g_n)$ to $\tilde{M}_n$. Let $\tilde{l}_n$, $\tilde{k}_n$, $\tilde{x}_n$, $\tilde{y}_n$ be lifts of $\bar{l}_n$, $\bar{k}_n$, $\bar{x}_n$ and $\bar{y}_n$ with $\tilde{x}_n \in \tilde{l}_n$ and $\tilde{x}_n \cup \tilde{y}_n = \partial\tilde{k}_n$. The point $\partial\tilde{l}_n \setminus \tilde{x}_n$ is the image of $\bar{x}_n$ under a covering transformation. Consider the $\tilde{m}_n$-geodesic arc $\tilde{l}'_n \subset \partial\tilde{C}(g_n)$ joining $\tilde{y}_n$ to its image under this covering transformation. The projection $\bar{l}'_n$ of $\tilde{l}'_n$ in $\bar{C}(g_n)$ is a $\bar{m}_n$-geodesic loop based at $\bar{y}_n$ that is homotopic to $\bar{l}_n$.

Since $\bar{x}_n$ and $\bar{y}_n$ lie in the simple closed geodesics $\bar{d}_n$ or $\bar{d}'_n$, they are uniformly far from the cone singularities (see Lemma 2.7). Hence $\bar{l}_n$ and $\bar{l}'_n$ are uniformly far from the cone singularities as well. It follows that there is a uniform neighbourhood of $\bar{l}_n$ which is disjoint from the cone singularities. Let $\eta > 0$ be such that $\mathcal{N}_n = \{\bar{x} \in (\bar{M}, \bar{g}_n) | d(\bar{x}, \bar{l}_n) \leq \eta\}$ is disjoint from the cone singularities.

Let $\tilde{z}_n$ be a point of $\partial\tilde{C}(g_n)$, one can locally define a support disc at $\tilde{z}_n$ namely a totally geodesic 2-dimensional disc that contains $\tilde{z}_n$ in its interior and is disjoint from the interior of $\tilde{C}(g_n)$. Notice that we can extend such a disc in any direction until it hits a singularity. Especially for a point $\tilde{z}_n \in \partial\tilde{C}(g_n) \cap \tilde{\mathcal{N}}_n$ we can consider a biggest (for the inclusion) local support disc $D_{\tilde{z}_n}$ at $\tilde{z}_n$ that lies in $\tilde{\mathcal{N}}_n$, namely $\partial D_{\tilde{z}_n} \subset \partial\tilde{\mathcal{N}}_n$. We have the following Claim:

Claim 3.6. *Let $\tilde{z}_n \in \tilde{l}_n$ be a sequence of points such that $D_{\tilde{z}_n}$ intersects $D_{\tilde{x}_n}$ for any n and let $\tilde{a}_n \subset \tilde{l}_n$ be the arc joining $\tilde{x}_n$ to $\tilde{z}_n$. Assume that $d(\tilde{x}_n, \tilde{z}_n)$ is bounded. Then $i(\tilde{a}_n, \lambda_n) \longrightarrow 0$.*

Proof. Since the length of $\tilde{k}_n$ tends to 0, $D_{\tilde{x}_n}$ and $D_{\tilde{y}_n}$ converge to each other. More precisely, consider the ball $B(\tilde{x}_n, R) \subset \tilde{M}_n$, then $B(\tilde{x}_n, R)$ converge in the pointed Gromov-Haudorff topology to a hyperbolic ball with cone singularities $B(\tilde{x}_\infty, R)$. Then $D_{\tilde{x}_n} \cap B(\tilde{x}_n, R)$ and $D_{\tilde{y}_n} \cap B(\tilde{x}_n, R)$ both converge to the same totally geodesic subspace $D_{\tilde{x}_\infty} \cap B(\tilde{x}_\infty, R)$ of $B(\tilde{x}_\infty, R)$.

In order to get a contradiction, let us assume that the dihedral angle between $D_{\tilde{z}_n}$ and $D_{\tilde{x}_n}$ is bounded away from 0. Since $d(\tilde{x}_n, \tilde{z}_n)$ is bounded, $D_{\tilde{z}_n} \cap B(\tilde{x}_n, R)$ converges to a totally geodesic subspace $D_{\tilde{z}_\infty} \cap B(\tilde{x}_\infty, R)$ of $B(\tilde{x}_\infty, R)$ for R large enough. Furthermore, $D_{\tilde{z}_\infty} \cap B(\tilde{x}_\infty, R)$ intersects $D_{\tilde{x}_\infty} \cap B(\tilde{x}_\infty, R) = D_{\tilde{y}_\infty} \cap B(\tilde{x}_\infty, R)$ and there the dihedral angle is not 0. It follows that, for n large enough, $D_{\tilde{z}_n}$ intersects $D_{\tilde{y}_n}$ transversely. This contradicts the convexity of $\tilde{C}(g_n)$. Thus we conclude that he dihedral angle between $D_{\tilde{z}_n}$ and $D_{\tilde{x}_n}$ tends to 0. By definition, this dihedral angle is bigger than $i(\tilde{a}_n, \lambda_n)$, the claim follows. □

Now let $\tilde{z}_n$ be the last point of $\tilde{l}_n$ such that $D_{\tilde{z}_n}$ intersects $D_{\tilde{x}_n}$, namely

- for any point $\tilde{w}_n \in \tilde{l}_n$ between $\tilde{x}_n$ and $\tilde{z}_n$, $D_{\tilde{w}_n}$ intersects $D_{\tilde{x}_n}$,
- there are points $\tilde{w}_n \subset \tilde{l}_n$ arbitrarily close to $\tilde{z}_n$ such that $U_{\tilde{w}_n}$ is disjoint from $U_{\tilde{x}_n}$.

Let us show that $d(\tilde{x}_n, \tilde{z}_n)$ tends to ∞.

Claim 3.7. *We have $d(\tilde{x}_n, \tilde{z}_n) \longrightarrow \infty$.*

Proof. Otherwise, extract a subsequence such that there exists R with $R > d(\tilde{x}_n, \tilde{z}_n)$ for any n. By Claim 3.6, $D_{\tilde{x}_n} \cap B(\tilde{x}_n, R)$ and $D_{\tilde{z}_n} \cap B(\tilde{x}_n, R)$ both converge to $D_{\tilde{x}_\infty} \cap B(\tilde{x}_\infty, R)$. Since $\tilde{z}_n$ lies in $\tilde{l}_n$, it is at a distance at least η from $\partial D_{\tilde{x}_n} \subset \partial \mathcal{V}_n$. It follows that $(\tilde{z}_n)_{n\in\mathbb{N}}$ converge to a point $\tilde{z}_\infty$ lying inside $D_{\tilde{x}_\infty} \cap B(\tilde{x}_\infty, R)$.

On the other hand there is a sequence of points $\tilde{w}_n \subset \tilde{l}_n$ converging to $\tilde{z}_\infty$ such that $D_{\tilde{z}_n}$ is disjoint from $D_{\tilde{x}_n}$. Since $D_{\tilde{z}_n} \cap B(\tilde{x}_\infty, R)$ also converges to $D_{\tilde{x}_\infty} \cap B(\tilde{x}_\infty, R)$, $D_{\tilde{w}_n}$ has to be close to the interior of both $D_{\tilde{x}_n}$ and $D_{\tilde{z}_n}$ for n large enough. Furthermore, by assumption, $D_{\tilde{w}_n}$ is disjoint from both $D_{\tilde{x}_n}$ and $D_{\tilde{z}_n}$. It is easy to see that such a situation can not happen for support discs. This contradiction allows to conclude that $d(\tilde{x}_n, \tilde{z}_n) \longrightarrow \infty$. ∎

The following lemma will allow us to show that a $\bar{m}_n$-geodesic loop based at $\bar{x}_n$ is almost not bend:

Lemma 3.8. *Consider a loop $\bar{l}_n$ on $C(g_n)$ based at $\bar{x}_n$, which is geodesic for $\bar{m}_n$ (except at $\bar{x}_n$). Let $\bar{f}_n \subset C(g_n)$ be the $\bar{g}_n$-geodesic loop based at $\bar{x}_n$ that is homotopic to $\bar{l}_n$ by a homotopy that fixes $\bar{x}_n$. Assume that $\ell_{\bar{g}_n}(\bar{f}_n)$ is bounded. Then the bending of $\bar{l}_n$ tends to* 0*, namely, $i(\bar{l}_n, \bar{\lambda}_n) \longrightarrow 0$.*

Proof. Denote $\tilde{z}_n = \partial\tilde{l}_n \setminus \tilde{x}_n$ the other endpoint of $\tilde{l}_n$. Since $\ell_{\tilde{f}_n}(\bar{g}_n)$ is bounded, it follows from Claim 3.7 that $D_{\tilde{z}_n}$ intersects $D_{\tilde{x}_n}$. By Claim 3.6, $i(\bar{l}_n, \bar{\lambda}_n) \longrightarrow 0$. ∎

Notice that Claim 3.8 still holds (with the same proof) if we replace $\bar{x}_n$ with $\bar{y}_n$.

Let us choose for $\bar{l}_n$ a shortest $\bar{m}_n$-geodesic loop based at $\bar{x}_n$. Since the area of $(\bar{S} \sqcup \bar{S}', \bar{m}_n)$ is bounded, there is a constant $Q > 0$ such that $\ell_{\bar{m}_n}(\bar{l}_n) \leq Q$. By Claim 3.8, we have $i(\bar{l}_n, \bar{\lambda}_n) \longrightarrow 0$. Let $\bar{l}'_n$ be the geodesic loop based at $\bar{y}_n$ that is homotopic to $\bar{k}_n^{-1}\bar{l}_n\bar{k}_n$ by a homotopy that fixes $\bar{y}_n$. Let $\bar{f}'_n \subset C(g_n)$ be the $\bar{g}_n$-geodesic loop based at $\bar{y}_n$ that is homotopic to $\bar{l}'_n$ by a homotopy that fixes $\bar{y}_n$. Since $\ell_{\bar{g}_n}(\bar{k}_n) \longrightarrow 0$, $\ell_{\bar{g}_n}(\bar{f}'_n) \leq 2\ell_{\bar{g}_n}(\bar{k}_n) + \ell_{\bar{m}_n}(\bar{l}_n)$ is bounded. By Claim 3.8, we have $i(\bar{l}'_n, \bar{\lambda}_n) \longrightarrow 0$. By construction, $\bar{l}_n$ and $\bar{l}'_n$ are freely homotopic in $\bar{C}(g_n)$. Hence there is an annulus $\bar{A}_n$ bounded by $\bar{l}_n$ to $\bar{l}'_n$.

If $\bar{y}_n$ lies in $\bar{d}'_n$, then $\bar{l}_n$ and $\bar{l}'_n$ lie in different components of $\partial\bar{M}$. In particular $\bar{A}_n$ is an essential annulus for any n. Furthermore, we have $i(\bar{\lambda}_n, \partial\bar{A}_n) \leq i(\bar{l}_n, \bar{\lambda}_n) + i(\bar{l}'_n, \bar{\lambda}_n) \longrightarrow 0$. Thus we have:

Claim 3.9. *If $\bar{y}_n$ lies in $\bar{d}'_n$ for infinitely many n, there is a sequence of essential annuli $\bar{A}_n$ such that, up to extracting a subsequence, $i(\bar{\lambda}_n, \partial\bar{A}_n) \longrightarrow 0$.*

If $\bar{y}_n$ lies in $\bar{d}_n$ then $\bar{A}_n$ is homotopic to an annulus $\bar{E}_n \subset \partial\bar{C}(g_n)$. It follows from the construction that $\bar{A}_n$ and $\bar{E}_n$ can be chosen so that $\bar{k}_n \subset \bar{A}_n$ and $\bar{\kappa}_n \subset \bar{E}_n$. Let us show that $\bar{E}_n$ is embedded in $\partial\bar{C}(g_n)$:

Claim 3.10. *The loops $\bar{l}_n$ and $\bar{l}'_n$ are disjoint.*

Proof. It follows from Claim 3.8 that $\bar{l}_n$ and $\bar{l}'_n$ are very close to being $\bar{g}_n$-geodesic loops. Since $\ell_{\bar{m}_n}(\bar{k}_n) \longrightarrow 0$, the $\bar{m}_n$-lengths of $\bar{l}_n$ and $\bar{l}'_n$ are very close. Notice that this holds for any $\bar{m}_n$-geodesic loop based at $\bar{x}_n$ with bounded length, namely the assumption that $\bar{l}_n$ is a shortest loop is not needed here. It follows that if there is a loop based at $\bar{y}_n$ that is shorter than $\bar{l}'_n$ then there is a loop based at $\bar{x}_n$ that is shorter than $\bar{l}_n$, up to a some $\varepsilon_n \longrightarrow 0$. Since $\bar{l}_n$ has been chosen to be a shortest loops based at $\bar{x}_n$, there is no loop based at $\bar{y}_n$ with a length smaller than $\ell_{\bar{m}_n}(\bar{l}_n) - \varepsilon_n$. Hence, up to moving $\bar{x}_n$ and $\bar{y}_n$ a little bit, we may assume that $\bar{l}_n$ and $\bar{l}'_n$ are shortest $\bar{m}_n$-geodesic loops based at $\bar{x}_n$ and $\bar{y}_n$.

Let $\bar{e}_n$ be the simple closed geodesic in the homotopy class of $\bar{l}_n$ and $\bar{l}'_n$. By the Margulis Lemma, there is $\varepsilon > 0$ such that there is an embedded annulus $\bar{E}'_n \subset \partial\bar{C}(g_n)$ around $\bar{e}_n$ with $\varepsilon \leq \ell_{\bar{m}_n}(\bar{e}'_n) \leq \ell_{\bar{m}_n}(\bar{l}_n)$ for any

component $\bar{e}'_n$ of $\partial \bar{E}'_n$. Let $\bar{v}_n$ be a shortest arc among the arcs connecting $\bar{x}_n$ to $\partial \bar{E}_n$ and let $\bar{w}_n$ be a shortest arc among the arcs connecting $\bar{y}_n$ to $\partial \bar{E}_n$. Let $\bar{e}'_n$, resp. $\bar{e}"_n$, be the component of $\partial \bar{E}_n$ that intersects $\bar{v}_n$, resp. $\bar{w}_n$ (we may have $\bar{e}'_n = \bar{e}"_n$). Since $\bar{l}_n$ and $\bar{l}'_n$ are shortest loop, $\bar{l}_n$ is homotopic to $\bar{v}_n \bar{e}'_n \bar{v}_n^{-1}$ and $\bar{l}'_n$ is homotopic to $\bar{w}_n \bar{e}"_n \bar{w}_n^{-1}$. It follows from the minimality of $\bar{v}_n$ and $\bar{w}_n$ that $\bar{v}_n$ and $\bar{w}_n$ are disjoint.

If $\bar{e}'_n = \bar{e}"_n$ then $\bar{\kappa}_n$ is homotopic to the concatenation of $\bar{v}_n$, $\bar{w}_n$ and an arc embedded in $\bar{e}'_n$. In particular $\ell_{\bar{m}_n}(\bar{\kappa}_n) \leq \ell_{\bar{m}_n}(\bar{v}_n) + \ell_{\bar{m}_n}(\bar{w}_n) + Q$. Since $\bar{v}_n$ and $\bar{w}_n$ are shortest arcs, there is $Q' > 0$ such that $\ell_{\bar{m}_n}(\bar{v}_n) \leq Q'$ and $\ell_{\bar{m}_n}(\bar{w}_n) \leq Q'$ (for example we can take Q' to be a bound on the diameter of the ε-thick part of $(\bar{S}, \bar{m}_n)$). Since $\ell_{\bar{m}_n}(\kappa_n) \longrightarrow \infty$, then $\bar{e}'_n \neq \bar{e}"_n$.

Thus we have proven that $\bar{v}_n$ and $\bar{w}_n$ are disjoint and that $\bar{e}'_n \neq \bar{e}"_n$. It follows that $\bar{l}_n$ and $\bar{l}'_n$ are disjoint. □

Since $\bar{l}_n$ and $\bar{l}'_n$ are disjoint, we can change $\bar{E}_n$ so that it is an embedded annulus.

Claim 3.11. *We have* $\liminf i(\bar{\kappa}_n, \bar{\lambda}_n) \geq \pi$.

Proof. The curve $\bar{k}_n \cup \bar{\kappa}_n$ is a skew polygon (up to approximating $\bar{\kappa}_n$ by piecewise geodesic segments) and bounds a disc in $\bar{C}(g_n)$. Consider the geodesic cone $\bar{D}_n$ from $\bar{x}_n$ to $\bar{t}_n \cup \bar{a}_n$. As in the proof of Lemma 3.2, the induced metric on $\bar{D}_n$ is a Riemannian metric with curvature bounded above by -1 and with cone singularities with cone angles of at least 2π. Since $\bar{k}_n$ is short, the edges of the piecewise geodesic segment $\bar{\kappa}_n$ that contains $\bar{x}_n$ and $\bar{y}_n$ have close tangent spaces at $\bar{x}_n$ and $\bar{y}_n$. It follows that the sum of the internal angles $\bar{D}_n$ at $\bar{x}_n$ and $\bar{y}_n$ is at least $\pi - \psi_n$ where ψ_n depends on $l_{\bar{g}_n}(\bar{t}_n)$ and satisfies $\psi_n \longrightarrow 0$. Now the Gauss-Bonnet formula shows that $\liminf i(\bar{\kappa}_n, \bar{\lambda}_n) \geq \pi$. □

We will deduce from this inequality that the homotopy class of $\bar{l}_n$ does not depend on n (up to extracting a subsequence), because otherwise its intersection with λ_n would blow up.

Claim 3.12. *There is a subsequence such that the free homotopy class of $\bar{l}_n$ in S does not depend on n.*

Proof. Fix a reference metric on S and denote by $\bar{e}_n$ the geodesic in the homotopy class of $\bar{l}_n$. Extract a subsequence such that $(\bar{e}_n)_{n \in \mathbb{N}}$ converges to a geodesic lamination $\bar{\mu}$ in the Hausdorff topology. Assume that $\bar{\mu}$ is not a simple closed curve. Consider a simple closed curve $\bar{f}$ that intersects transversely a minimal sublamination of $\bar{\mu}$ and does not intersect $\bar{d}_n$ transversely. Let $\bar{f}_n \subset \partial \bar{C}(g_n)$ be the geodesic representative of $\bar{f}$. Since $\bar{\mu}$ is not a simple closed curve, $\sharp\{\bar{f}_n \cap \bar{e}_n\} \longrightarrow \infty$. Since $\bar{f}_n$ is disjoint from $\bar{d}_n$, any times $\bar{f}_n$ crosses the annulus $\bar{E}_n$ an amount of at least

$$i(\bar{\kappa}_n, \bar{\lambda}_n) - i(\bar{l}_n, \bar{\lambda}_n) - i(\bar{l}'_n, \bar{\lambda}_n)$$

contributes to $i(\bar{f}, \bar{\lambda}_n)$. It follows that

$$i(\bar{f}, \bar{\lambda}_n) \geq \sharp\{\bar{f}_n \cap \bar{l}_n\}(i(\bar{\kappa}_n, \bar{\lambda}_n) - i(\bar{l}_n, \bar{\lambda}_n) - i(\bar{l}'_n, \bar{\lambda}_n)) \ .$$

Since $i(\bar{l}_n, \bar{\lambda}_n) \longrightarrow 0$ (Claim 3.8) and $\liminf i(\bar{\kappa}_n, \bar{\lambda}_n) \geq \pi$ (Claim 3.11), we obtain $i(\bar{f}, \bar{\lambda}_n) \longrightarrow \infty$. This contradicts the assumption that $\bar{\lambda}_n$ converge to $\bar{\lambda}_\infty$. Now we conclude that $\bar{\mu}$ is a simple closed curve. Hence there is a subsequence such that the free homotopy class of $\bar{l}_n$ in S does not depend on n. □

We can now prove that if $\bar{y}_n$ lies in $\bar{d}_n$ then $\bar{\lambda}_\infty$ contains a leaf with a weight equal to at least π.

Claim 3.13. *If $\bar{y}_n$ lies in $\bar{d}_n$ then $\bar{\lambda}_\infty$ contains a leaf with a weight equal to at least π.*

Proof. Let $\bar{e}$ be the free homotopy class of $\bar{l}_n$. Now, using the same arguments as in the proof of Claim 3.12, we can show that $\bar{e}$ is a leaf of $\bar{\lambda}_\infty$ with a weight greater than or equal to π. Anytime $\bar{d}_n$ intersects $\bar{l}_n$, an amount of at least

$$i(\bar{a}_n, \lambda_n) - i(\bar{l}_n, \bar{\lambda}_n) - i(\bar{l}'_n, \bar{\lambda}_n)$$

contributes to $i(\bar{d}_n, \bar{\lambda}_n)$. It follows then from Lemma 3.8 and Claim 3.11 that $\bar{e}$ is a leaf of $\bar{\lambda}_\infty$ with weight of at least π. □

It is now easy to conclude the proof of Lemma 3.4. Under the assumptions of Lemma 3.4, namely when there is a simple closed curve d such that $\ell_{m_n}(d_n) \longrightarrow \infty$, it follows from Claims 3.5, 3.9 and 3.13 that either there is a sequence of essential annuli $\bar{A}_n \subset \bar{C}(g_n)$ such that $i(\bar{\lambda}_n, \partial\bar{A}_n) \longrightarrow 0$ or $\bar{\lambda}_\infty$ contains a leaf with a weight equal to at least π. If there is a sequence of essential annuli $\bar{A}_n$ we consider the projection A_n of $\bar{A}_n$. Although A_n may not be embedded, it follows from the Annulus Theorem [Wal67] that any neighbourhood of A_n contains an embedded annulus E_n. We have then $i(\lambda_n, \partial E_n) \longrightarrow 0$. If $\bar{\lambda}_\infty$ contains a leaf $\bar{e}$ with a weight equal to at least π, then $\bar{e}$ projects to a leaf $e \subset \partial M$ of λ_∞ with a weight equal to at least π. $\square$

We can now deduce from Lemma 3.4 that under the assumptions of Lemma 3.1, the sequence of induced metrics $(m_n)_{n\ni\mathbb{N}}$ on $\partial C(g_n)$ is bounded.

Lemma 3.14. *Under the assumptions of Lemma 3.1, the sequence of induced metrics $(m_n)_{n\in\mathbb{N}}$ on $\partial C(g_n)$ is bounded.*

Proof. If (m_n) is unbounded, then there is a simple closed curve $d \subset S$ and a geodesic representative $d_n \subset \partial C(g_n)$ such that $l_{m_n}(d_n)$ is unbounded. By Lemma 3.4 and the assumptions on λ, there is a sequence E_n of essential annuli such that $i(\lambda_n, \partial E_n) \longrightarrow 0$. By [Lec06, Lemme C5] the existence of such a sequence of annuli contradicts the assumption: $i(\lambda_\infty, \partial A) \geq \eta$ for any essential annulus A. $\square$

3.5 Convergence of convex cores

The last step in the proof of Lemma 3.1 is to show that, under the assumption that the sequence of metrics on the boundary are bounded, a subsequence of convex cores converges for the bilipschitz topology.

Lemma 3.15. *Let $N = S \times I$, let $p_1, \cdots, p_{n_0}$ be distincts points on S, and let $\kappa_i = \{p_i\} \times I, 1 \leq i \leq n_0$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of hyperbolic metrics on N with cone singularities at the κ_i, such that, for all n, (N, g_n) is a convex co-compact manifold with particles. Let (m_n^+, m_n^-) be the metrics induced by m_n on each component of $\partial C(N)$. Suppose that the sequences $\{m_n^\pm\}$ converge. Then, after taking a subsequence, (g_n) converges to a convex co-compact hyperbolic metric with particles.*

Proof. First notice that since the cone angles are less than π, there is by Lemma 2.7 a positive lower bound for the distance between two components of the singularity locus. Consider a point $x_n \in C(g_n)$, extract a subsequence such that the sequence $(x_n, C(g_n))$ converges in the Hausdorff topology (such a subsequence always exists). By [BP01, Proposition 3.2.6], the limit $(x_\infty, C(g)_\infty)$ is a hyperbolic manifold with cone singularities. By [BP01, Proposition 3.3.1], the sequence $(x_n, C(g_n))$ converges to $(x_\infty, C(g)_\infty)$ in the bilipschitz topology. It follows easily that $C(g)_\infty$ is a convex manifold with cone singularities. The bilipschitz convergences implies that the cone singularities of the limit manifold are orthogonal to the boundary.

Since $(x_\infty, C(g)_\infty)$ converges in the bilipschitz topology, if the diameter of $C(g_n)$ is uniformly bounded, then the injectivity radius is bounded from below outside a small neighborhood of the cone singularities. This implies that no collapse occurs in the limit and that, for n large enough, $C(g)_\infty$ is homeomorphic to $C(g_n)$.

Lemma 3.16. *The diameter of $C(g_n)$ is uniformely bounded.*

Proof. As we have said before the branched cover $(\bar{M}, \bar{g}_n)$ can be viewed as a manifold with negative curvature with concentrated negative curvature along the singularities. It follows that $(\bar{M}, \bar{g}_n)$ satisfies the Margulis Lemma. As a consequence a very short geodesic in $(\bar{M}, \bar{g}_n)$ lies in a very deep embedded tube. Using that observation we will show that there is a uniform lower bound on the length of any fixed curve in $\bar{C}(g_n)$.

Claim 3.17. *Let $\bar{c} \subset \bar{S}$ be a simple closed curve. Then there is $Q > 0$ such that if $\bar{c}_n \subset \bar{C}(g_n)$ denotes the geodesic representative of $\bar{c}$, $l_{\bar{g}_n}(\bar{c}_n) \geq Q$ for any $n \in \mathbb{N}$.*

Notice that Q does not depend on n.

Proof. Assume the contrary, that is (after extracting a subsequence), $\lim l_{g_n}(c_n) = 0$. Then $\bar{c}_n$ is the core of a deeper and deeper Margulis tube $\bar{T}_n$. Notice that since (m_n^+, m_n^-) converges, there is no short curve in $\partial\bar{C}(g_n)$, more precisely, there is a uniform lower bound on the length of any simple closed geodesic in $\partial\bar{C}(g_n)$. It is

a standard property of pleated surfaces that if such a pleated surface penetrates deep inside a Margulis tube, it does so along a long and thin annuli. More precisely, given $\varepsilon > 0$ small enough, there is $\eta(\varepsilon)$ such that $\partial\bar{C}(g_n) \cap (\bar{M}, \bar{g}_n)^\varepsilon \subset \partial\bar{C}(g_n)^\eta$ (see the proof of [Min99, Lemma 6.3], the proof extends easily to manifolds with curvature ≤ -1). Thus taking ε small enough so that $\eta(\varepsilon)$ is less than the length of the shortest curve in $\partial\bar{C}(g_n)$, and taking $\bar{T}_n$ to be an ε-Margulis tube, we get that $\bar{T}_n$ lies entirely in the interior of $\bar{C}(g_n)$. Consider a simple closed curve $d \subset \bar{S}$ that intersects $\bar{c}$ essentially. By Lemma 3.2, there is an essential annulus $A_n \subset C(g_n)$ which is in the homotopy class defined by $d \times I$ such that the area of A_n is at most $\ell_{m_n^+}(d_+) + \ell_{m_n^-}(d_-)$. In particular, since the sequences $\{m_n^\pm\}$ converge, the area of A_n is bounded. On the other hand, since d intersects c essentially, A_n intersects c_n essentially. In particular, A_n intersects T_n along a disc D_n. When the length of c_n tends to 0, $d(c_n, \partial T_n) \longrightarrow \infty$ (see [Min99, Lemma 6.1]). It follows that the diameter of D_n, and hence its area, tends to ∞ when the length of c_n tends to 0. Thus an upper bound for the area of $A_n \supset D_n$ yields a lower bound for the length of c_n. This concludes the proof of Claim 3.17. ☐

Consider now two simple closed curve $c, d \subset S$ such that the components of $S \setminus (c \cup d)$ are discs. Two such curves are said to fill the surface S. Consider essential annuli A_n and B_n in $C(g_n)$ in the homotopy classes defined by c and d, constructed as in Lemma 3.2. In particular A_n and B_n have bounded area. Since A_n and B_n have bounded areas and negative curvature, the only way for them to have a large diameter is to have a very short core curve. but this would contradict Claim 3.17. Thus we can conclude that A_n and B_n have uniformely bounded diameters.

Let $\mathcal{B}_{1\leq k\leq p}$ be the closure of the components of $S \times I \setminus (c \times I \cup d \times I)$. Our manifold $N = S \times I$ is the union of the $\mathcal{B}_k$ and the $\mathcal{B}_k$ are all balls. Define a surjective map $f_n : S \times I \to C(g_n)$ that maps $c \times I$ and $d \times I$ to A_n and B_n respectively and such that the restriction of f_n to each $\mathcal{B}_k$ is an immersion. For each k, the image of $\partial\mathcal{B}_k$ lies in $A_n \cup B_n \cup \partial C(g_n)$. Since A_n and B_n have bounded diameters and since the induced metric on $\partial C(g_n)$ is bounded, the diameter of $f_n(\partial\mathcal{B}_k)$ is bounded for any k. It follows that $f_n(\mathcal{B}_k)$ has a bounded diameter for any k. Since f_n is surjective, this implies that $C(g_n)$ has a uniformly bounded diameter. ☐

As was noticed before Lemma 3.16, the conclusion of Lemma 3.15 follows from Lemma 3.16 and the fact that $(x_n, C(g_n))$ converges to $(x_\infty, C(g_n))$ in the bilipschitz topology. ☐

3.6 The bending lamination of the convex core

To finish the proof of Lemma 3.1 we only have to check that the induced bending lamination on the boundary of the convex core of the limit manifold is the limit of the bending laminations. We can state the result as follows.

Lemma 3.18. *Let $N = S \times I$, let $p_1, \cdots, p_{n_0}$ be distincts points on S, and let $\kappa_i = \{p_i\} \times I, 1 \leq i \leq n_0$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of quasifuchsian metrics on N with a particle of angles θ_n^i along κ_i, $1 \leq i \leq n_0$. Let λ_n be the measured bending laminations on the boundary of the convex core of N (considered as a measured lamination on ∂N minus the endpoints of the κ_i), and let m_n be the induced metric on the boundary of N. Suppose that (g_n) converges in bilipschitz topology towards a quasifuchsian metric with particles g, with cone angles $\theta^i \in (0, \pi)$ along κ_i. Then $(m_n)_{n\in\mathbb{N}}$ converges to the induced metric m on the boundary of the convex core in (N, g), while $(\lambda_n)_{n\in\mathbb{N}}$ converges to the measured bending lamination λ of the boundary of the convex core of (N, g).*

Proof. We consider a above the finite cover $\bar{N}$ of N ramified along the cone singularities, chosen so that all cone angles in $\bar{N}$ have angle larger than 2π. This is useful below since we will use negative curvature arguments, in particular the existence of a geodesic segment in a homotopy class with fixed endpoints. Clearly it is sufficient to prove the lemma for $\bar{N}$, where the "convex core" considered is $\bar{C}(g_n)$, the lift to $\bar{N}$ of $C(g_n)$, since once the result is obtained in $\bar{N}$, we can take the quotient by the group of deck transformations to obtain the result on N.

Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of segments in $\bar{N}$, with γ_n geodesic for $\bar{g}_n$ for all $n \in \mathbb{N}$. Suppose that $(\gamma_n)_{n\in\mathbb{N}}$ converges to a segment γ. We know that $\bar{g}_n \to \bar{g}$ in the bilipschitz topology and, in hyperbolic geometry, any segment which is close to realizing the distance between its endpoints is close to a geodesic segment. So γ is

geodesic for $\bar{g}$. Conversely, any geodesic segment for $\bar{g}$ is a Haudorff limit of geodesic segments for the g_n. The same holds for closed geodesics.

Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of $\bar{N}$ such that, for all $n\in\mathbb{N}$, Ω_n is convex for $\bar{g}_n$. Suppose that $\Omega_n\to\Omega$ in the Hausdorff topology. The definition of a convex subset and the previous paragraph show that Ω is convex, since any geodesic segment γ in $(\bar{N},\bar{g})$ with endpoints in the interior of Ω is the limit of a sequence of geodesic segments γ_n, with γ_n geodesic for $\bar{g}_n$. Since γ_n has endpoints in Ω_n (for n large enough) and Ω_n is convex for $\bar{g}_n$, $\gamma_n\subset\Omega_n$, and therefore $\bar{\gamma}\subset\Omega$, and Ω is convex for $\bar{g}$. Conversely, a similar argument shows that any compact convex subset for $\bar{g}$ is the Hausdorff limit of a sequence of compact convex subsets of the metrics $\bar{g}_n$.

For all n, $\bar{C}(g_n)$ contains all closed geodesics in $(\bar{N},\bar{g}_n)$. Given a non-trivial homotopy class α in $\bar{N}_r$ (the complement of the singular curves in $\bar{N}$), it is realized for each $n\in\mathbb{N}$ by a (unique) closed geodesic γ_n in $(\bar{N},\bar{g}_n)$, and the sequence $(\gamma_n)_{n\in\mathbb{N}}$ converges to the closed geodesic γ which realizes α in $(\bar{N},\bar{g})$. For each $n\in\mathbb{N}$, $\gamma_n\subset\bar{C}(g_n)$. Moreover we have seen that the diameter of the $\bar{C}(g_n)$ is bounded. It follows that $(\bar{C}(g_n))_{n\in\mathbb{N}}$ converges – after extracting a subsequence – to a limit subset C' which contains all closed geodesics in $(\bar{N},\bar{g})$.

Since C' is the limit of a sequence of convex subset of the $(\bar{N},\bar{g}_n)$, it is convex. Moreover if $\Omega\subset C'$ is convex, then it is the limit of a sequence of convex subsets $\Omega_n\subset\bar{C}(g_n)$. But then $(\Omega_n\cap\bar{C}(g_n))_{n\in\mathbb{N}}$ is a sequence of convex subsets converging to Ω. Because the $\bar{C}(g_n)$ are minimal convex subsets, $\Omega_n\cap\bar{C}(g_n)=\bar{C}(g_n)$ for all n, so that $\Omega=C'$. So $C'=\bar{C}(g)$. This shows that $\bar{C}(g)$ is the Hausdorff limit of the $\bar{C}(g_n)$.

Note that it is not clear at this point whether $\partial\bar{C}(g_n)\to\partial\bar{C}(g)$ in the C^1 topology. However, a general fact is that, if $\phi:S\to H^3$ is a smooth embedding of a surface, and if $(\phi_n)_{n\in\mathbb{N}}$ is a sequence of Lipschitz embeddings of S in H^3 which converges to ϕ in the C^0 topology, then the distance d_n induced on S by the ϕ_n are larger, in the limit, than the distance d induced by ϕ:

$$\forall x,y\in S, \limsup_{n\to\infty} d_n(x,y)\geq d(x,y) \tag{1}$$

The same holds if ϕ is Lipschitz with locally convex image rather than smooth, see [AZ67]. Moreover, in case of equality in Equation (1), then the convergence of (ϕ_n) to ϕ is stronger, in the sense that the tangent plane to $\phi_n(S)$ almost everywhere converges to the tangent plane to $\phi(S)$.

Coming back to $\partial\bar{C}(g_n)$, the C^0 convergence towards $\partial\bar{C}(g)$ (together with the bilipschitz convergence of $\bar{g}_n$ to $\bar{g}$) is sufficient to insure that the metric $\bar{m}_n$ on $\partial C(g_n)$ is *larger* in the limit than the metric $\bar{m}$ induced by $\bar{g}$ on $\partial\bar{C}(g)$. In other terms, if $x,y\in\partial\bar{C}(g)$ and $x_n,y_n\in\partial\bar{C}(g_n)$ are such that $\lim x_n=x, \lim y_n=y$, then there exists for each $\epsilon>0$ some $N_0\in\mathbb{N}$ such that, for all $n\geq N_0$,

$$d_{\bar{m}_n}(x_n,y_n)\geq(1-\epsilon)d_{\bar{m}}(x,y)\ .$$

It follows that the lengths of the closed geodesics in S for $\bar{m}_n$ are bounded from below by $(1-\epsilon)$ times their lengths for $\bar{m}$.

Since the metrics $\bar{m}_n$ and $\bar{m}$ are hyperbolic metrics with cone singularities of fixed angles, this shows, using standards arguments based for instance on pants decompositions, that $\bar{m}_n\to\bar{m}$ (see Section A.2, or [DP07]).

It then follows that as $n\to\infty$, $\bar{m}_n$ is bounded from above by $(1+\epsilon)\bar{m}$: under the same hypothesis as above,

$$d_{\bar{m}_n}(x_n,y_n)\leq(1+\epsilon)d_{\bar{m}}(x,y)\ . \tag{2}$$

Since $\bar{m}_n\to\bar{m}$ in this sense of Equation (2), and moreover $\partial\bar{C}(g_n)\to\partial\bar{C}(g)$ in the C^0 topology, it follows that the convergence is actually stronger, and the tangent plane to $\partial\bar{C}(g_n)$ converges almost everywhere to the tangent plane to $\partial\bar{C}(g)$ (both exist almost everywhere by convexity). This implies that the measured laminations λ_n of the ∂C_n converge to λ. □

4 Prescribing the measured bending lamination on the boundary of the convex core

The goal of this section is to prove Theorem 1.12 and then Theorem 1.13. The proof of Theorem 1.12 is largely based on a well-known doubling argument already used for non-singular manifolds, which reduces the

infinitesimal rigidity with respect to the measured lamination (when the support of the lamination is along closed curves) to a rigidity statement proved by Hodgson and Kerckhoff [HK98] for hyperbolic cone-manifolds.

Theorem 1.13 is then a consequence, using the compactness statement proved in section 3.

4.1 A doubling argument

Let M be convex co-compact manifold with particles, and let $C(M)$ be its convex core. Suppose that the support of the measured bending lamination of $C(M)$ is a disjoint union of closed curves.

Definition 4.1. *The* **doubled convex core** *of M is the 3-dimensional hyperbolic manifold with cone singularities $DC(M)$ obtained by gluing two copies of $C(M)$ isometrically using the identification of their boundaries.*

We have seen that the singular locus of M does not intersect the support of the bending lamination on the boundary of the convex core – actually it even remains at a distance which is bounded from below by a constant depending only on the cone angles. So the "particles" intersect the boundary of the convex core inside faces, and moreover it does so orthogonally. It follows that the singular locus of $DC(M)$ is a disjoint union of closed curves, which are of two types:

- each "particle" p of M corresponds to a cone singularity along a closed curve in $DC(M)$, of length equal to twice the length of the intersection of p with $C(M)$,
- each closed curve in the support of the measured bending lamination of the boundary of $C(M)$ corresponds to a closed curve (of the same length) in $DC(M)$.

Still by definition, $DC(M)$ admits an isometric involution – exchanging the two copies of $C(M)$ which are glued to obtain $DC(M)$ – and the set of fixed points of this involution is a (non connected)) closed surface S, which corresponds to the boundaries of both copies of $C(M)$. This surface is orthogonal to the singularities of the first kind, and contains the singularities of the second kind.

4.2 Local deformations

The doubling trick explained above leads directly to a rigidity statement. We consider again a convex co-compact manifold M with particles, for which the measured bending lamination of the convex core is along closed curves $\gamma_1, \cdots, \gamma_N$, for which the bending angles are equal to $\alpha_1, \cdots, \alpha_N \in (0, \pi)$. As in the introduction, we call $\theta_1, \cdots, \theta_{n_0}$ the cone angles at the "particles", and let $\theta = (\theta_1, \cdots, \theta_{n_0})$.

Lemma 4.2. *There exists a neighborhood U of $(\alpha_1, \cdots, \alpha_N)$ in $(0, \pi)^N$ and a neighborhood V of the hyperbolic metric g on M in $\mathcal{QF}_{S,n_0,\theta}$ such that, if $(\alpha'_1, \cdots, \alpha'_N) \in U$, there is a unique $g' \in V$ for which the support of the measured bending lamination on $C(M)$ is $\alpha_1 \cup \cdots \cup \alpha_N$ and the bending angle on γ_i is α'_i, $1 \leq i \leq N$.*

Proof. Hodgson and Kerckhoff [HK98] proved a local deformation result for hyperbolic cone-manifolds. It follows from their result that there exists a unique cone-manifold close to $DC(M)$ with the same topology as $DC(M)$ (including the singular locus), the same angles at the cone singularities corresponding to the particles in M, and angles $2\alpha'_1, \cdots, 2\alpha'_N$ instead of $2\alpha_1, \cdots, 2\alpha_N$ at the cone singularities corresponding to the pleating lines of $C(M)$.

The uniqueness of D' shows that it has the same symmetry as $DC(M)$, that is, it admits an isometric involution fixing a surface S' isotopic to the surface S fixed by the isometric involution on $DC(M)$. By an easy symmetry argument, the cone singularities in D' corresponding to the particles in M still have to be orthogonal to S', while those corresponding to the pleating lines of $\partial C(M)$ have to be contained in S' (see [BO04, Section 8] for details on the uniqueness part of this argument).

Therefore, D' is the double of a hyperbolic manifold with convex boundary (obtained as the metric completion of one half of the complement of S' in D') with cone singularities orthogonal to the boundary. The boundary of this manifold is convex with no extremal point, so that it is the convex core of a quasifuchsian manifold with particles M', with the same cone angle as M at the particles and such that $\partial C(M')$ is pleated along the same lines as $\partial C(M)$, but with pleating angles $\alpha'_1, \cdots, \alpha'_N$ instead of $\alpha_1, \cdots, \alpha_N$.

The uniqueness of such a manifold, in the neighborhood of M, follows from the uniqueness of D' in the neighborhood of $DC(M)$. □

4.3 Proof of Theorem 1.12

Let $\gamma_1, \cdots, \gamma_N$ be the curves in the support of λ, considered as curves in ∂N. Following the doubling construction above, we define a closed manifold $D(N)$ by gluing two copies of N along their boundary. $D(N)$ contains two families of curves, which we still call $c_1, \cdots, c_{n_0}$ (corresponding to the particles in N) and $\gamma_1, \cdots, \gamma_N$ (corresponding to the pleating lines on the boundary of the convex core).

Let $\theta'_1, \cdots, \theta'_{n_0} \in (0, \pi)$ and $\alpha'_1, \cdots, \alpha'_N \in (0, \pi)$ be chosen such that:

- for all $i \in \{1, \cdots, n_0\}$, $0 \leq \theta'_i \leq \theta_i$, and $\theta'_i = \pi/k_i$ for some $k_i \in \mathbb{N}$,
- for all $j \in \{1, \cdots, N\}$, $0 \leq \alpha'_j \leq \alpha_j/2$, and $\alpha'_j = \pi/2l_j$ for some $l_j \in \mathbb{N}$.

The Orbifold Hyperbolization Theorem for cyclic orbifolds (initially stated by Thurston, and proved in [BP01, CHK00]) can be applied to show that there is a unique hyperbolic orbifold structure on $D(N)$ with singularities of angles θ'_i on the c_i and $2\alpha'_j$ on the γ_j.

Since the θ_i are in $(0, \pi)$, the results of Kojima [Koj98] shows that this orbifold structure can be deformed to a unique cone-manifold structure, with cone angles θ_i on the curves c_i and $2\alpha'_j$ on the curves γ_j.

Let $(\alpha_t)_{t\in[0,1]} = (\alpha_{1,t}, \cdots, \alpha_{N,t})_{t\in[0,1]}$ be the 1-parameter family defined by

$$\alpha_{j,t} = (1 - t)\alpha'_j + t\alpha_j,\ 1 \leq j \leq N\ .$$

Then for all $j \in \{1, \cdots, N\}$, $\alpha_{j,0} = \alpha'_j, \alpha_{j,1} = \alpha_j$. Let $I \in [0, 1]$ be the maximal interval containing 0 such that, for all $t \in I$:

- there exists a hyperbolic structure on $D(N)$ with cone singularities of angle θ_i on c_i, $1 \leq i \leq n_0$, and a cone singularity of angle $2\alpha_{j,t}$ on γ_j, $1 \leq j \leq N$,
- this hyperbolic structure has an isometric involution exchanging the two copies of N glued to obtain $D(N)$.

By construction, $I \neq \emptyset$. Lemma 4.2 shows that I is open, while Lemma 3.1 shows that I is closed. So $I = [0, 1]$, this proves the existence part of the statement because $D(N)$ with the hyperbolic cone-structure for $t = 1$ is obtained by doubling the convex core of a convex co-compact hyperbolic manifold with particles of angles θ_i and pleating angles α_i on the boundary, as needed.

For the uniqueness, the same deformation argument can be used to start from a cone-manifold structure on $D(N)$ and decrease the angles along the curves γ_j, $1 \leq j \leq N$, from $2\alpha_j$ to $2\alpha'_j$. Lemma 4.2 shows that the corresponding deformation of the hyperbolic cone-manifold structure exists and is unique. Since the endpoint of the deformation is unique (by the Orbifold Hyperbolization Theorem) there can be only one cone-manifold structure on $D(N)$ with angles θ_i on the curves c_i, $1 \leq i \leq n_0$, angle $2\alpha_j$ on the curve γ_j, $1 \leq j \leq N$, and the necessary symmetry property.

4.4 Proof of Theorem 1.13.

Given $\lambda_-, \lambda_+ \in \mathcal{ML}_{S,x}$ satisfying the hypothesis of Theorem 1.13, both are limits of a sequence of measured laminations $(\lambda_{-,n})_{n\in\mathbb{N}}, (\lambda_{+,n})_{n\in\mathbb{N}}$ with support along a union of closed curves, which satisfy the hypothesis of Theorem 1.12.

For all n, Theorem 1.12 shows that $\lambda_{-,n}$ and $\lambda_{+,n}$ are the upper and lower measured bending laminations of the boundary of the convex core for a unique quasifuchsian hyperbolic structure with particles g_n on $S \times \mathbb{R}$. Lemma 3.1, applied to this sequence of hyperbolic structures, shows that it has a subsequence which converges to a quasifuchsian hyperbolic structure with particles, for which the lower and upper measured bending laminations of the boundary of the convex core are λ_- and λ_+, respectively.

4.5 The conditions are necessary

Finally we check here that the hypothesis in Theorem 1.13 are necessary. It obviously follows that the hypothesis in Theorem 1.12 are also necessary.

Lemma 4.3. *Let M be a non Fuchsian quasifuchsian manifold with particles, let λ be the measured bending lamination on the boundary of its convex core. Then λ satisfies the hypothesis of Theorem 1.13.*

Proof. The hypothesis that the weight of each closed curve in the support of λ_- and λ_+ is less than π is clearly a consequence of the fact that $C(M)$ is convex and compact.

Suppose by contradiction that λ_- and λ_+ do not fill S. There exists then a sequence $(c_n)_{n\in\mathbb{N}}$ of simple closed curve in S such that

$$i(\lambda_-, c_n) + i(\lambda_+, c_n) \to 0 \ .$$

Let c_n^- and c_n^+ be the geodesic representatives of c_n in the lower and upper boundary components of $C(M)$, respectively.

Let $\bar{c}_n^-$ and $\bar{c}_n^+$ be lifts of c_n^- and c_n^+, respectively, to $\bar{M}$, corresponding to the same lift of c. Lemma 3.2 shows that there exists an annulus $A_n \subset \bar{M}$ bounded by $\bar{c}_n^-$ and $\bar{c}_n^+$ on which the induced metric is hyperbolic with cone points of negative singular curvature (cone angle larger than 2π). Moreover, the boundary of A_n is convex (for the induced metric) and its total curvature goes to 0 as $n \to \infty$. The Gauss-Bonnet formula then implies that the area of A_n goes to 0 as $n \to \infty$. Since the lengths of the $\bar{c}_n^-$ and $\bar{c}_n^+$ are bounded from below, this means that the distance between $\bar{c}_n^-$ and $\bar{c}_n^+$ in A_n goes to 0 as $n \to \infty$. Therefore, the distance between the upper and lower boundary of $C(M)$ is zero, a contradiction because we have supposed that M is not Fuchsian. $\square$

5 Earthquakes estimates

In this section we consider a convex co-compact manifold with particles M. The arguments in this more general case are the same as in the specific situation of quasifuchsian manifolds with particles. Its boundary ∂M has a number of marked points $x_1, \cdots, x_{n_0}$ which are the endpoints of the "particles", and to each is attached an angle $\theta_k \in (0, \pi)$, $1 \leq k \leq n$, which is the angle at the corresponding particles.

We identify ∂M with the boundary of its convex core (see Lemma 2.5). We will use the following notations.

- λ is the measured bending lamination of the boundary of the convex core.
- m is its induced metric.
- t is the (unique) hyperbolic metric in the conformal class at infinity τ, with cone singularities of prescribed angle θ_k at the marked point x_k.
- $G_\lambda(m)$ is the metric obtained by grafting the hyperbolic metric m along the measured lamination λ, so that $G_\lambda(m)$ has curvature in $[-1, 0]$. If for instance λ is rational, then $G_\lambda(m)$ is obtained by inserting a flat annulus in $(\partial M, m)$ for each closed curve in the support of λ, see e.g. [Dum08].

This section contains a basic estimates relating t to m. It will be useful in proving the compactness of a certain map and Theorem 1.7. Its statement is based on the following extension to hyperbolic surfaces with cone singularities of Thurston's Earthquake Theorem (as found in [Thu86b, Ker83]).

Theorem 5.1 ([BS06]). *For any $h, h' \in \mathcal{T}_{\Sigma,x,\theta}$, there is a unique measured lamination $\nu \in \mathcal{ML}_{\Sigma,x}$ such that the right earthquake along ν sends h to h'.*

The main estimate proved in this section, and the main tool for the proof of Theorems 1.7 and 1.14, is the following.

Proposition 5.2. *There exists a constant $C > 0$ (depending only on the topology of M) such that, if $\nu \in \mathcal{ML}_{\partial M,x}$ is the measured lamination such that $t = E_r(\nu)(m)$, then the length $L_m(\nu)$ is at most C.*

It is proved in Section 5.4, after some preliminary considerations. It is used below in Section 6.

5.1 The average curvature of geodesics

In this part we prove a technical statement which is useful at several points below. It is an extension to convex co-compact manifolds with particles of a result proved earlier by Bridgeman [Bri98] for the convex core of non-singular convex co-compact manifolds, or more generally of pleated surfaces in H^3. However the argument used here is inspired by Bonahon and Otal [BO04].

We consider a quasifuchsian manifold with particles, M, and call $\theta_1, \cdots, \theta_{n_0}$ the cone angles at the particles. By definition, $\theta_1, \cdots, \theta_{n_0} \in (0, \pi)$. Here S is one of the connected components of $\partial C(M)$.

Proposition 5.3. *There exists a constant $C_0 > 0$ such that, if γ is a geodesic segment on S transverse to λ, $i(\gamma, \lambda) \leq C_0(l_m(\gamma) + 1)$.*

Note that C_0 depends on the θ_i (at least the argument we use here does depend on the maximum of the θ_i) but not otherwise on M.

Proposition 5.3 will follow from the following lemma.

Lemma 5.4. *There exists $\lambda_1 > 0$ such that if γ is at distance at least $\epsilon_0/2$ from the intersection of S with the singular set of M and if $l_m(\gamma) \leq \epsilon_0/4$, then $i(\lambda, \gamma) \leq \lambda_1$.*

The proof of this lemma is based on some intermediate steps. The first is a consequence of Lemma 2.7.

Claim 5.5. *There exists $\rho_0 > 0$ such that any point in S at distance at least ϵ_0 in S from the singular points of S is also at distance at least ρ_0 in M from the singular set of M.*

Proof. Let $x \in S$ which is at distance at least ϵ_0 from the singular points of S, suppose that it is at distance strictly less than ϵ_0 from a particle p. Let y be a point in p which is closest from x, and let D be the totally geodesic disk of radius ϵ_0 orthogonal to p at y. This disk does not encounter any other particle by the second part of Lemma 2.7. Moreover $x \in D$ because y is at minimal distance from x among the points of p. We can therefore apply the third point in Lemma 2.7 to D, with Ω equal to the intersection of D with the convex core of M. The result follows. □

Corollary 5.6. *Let $x \in S$ be contained in the support of the bending lamination λ, and let D' be the totally geodesic disk of radius ρ_0 in M orthogonal to λ at x. Then D' does not intersect the singular set of M.*

After taking ρ_0 smaller if necessary, we have another simple statement which will be necessary below.

Claim 5.7. *Let $y \in \lambda$ be a point in the connected component of x in the intersection of S with D', and let g_y be the geodesic segment in the support of λ centered at y and of length $2\epsilon_0$. Then the angle between g_y and D' at y is at least $\pi/4$.*

Proof. We call g_x the geodesic segment contained in the support of λ centered at x and of length $2\epsilon_0$. g_x is disjoint from g_y on S while x is at distance at most ρ_0 from y, it follows that there exists $c > 0$ (depending on ϵ_0 and ρ_0, and going to 0 as $\rho_0 \to 0$ for fixed ϵ_0) such that the distance to g_x in S of any point of $g_y([-\epsilon_0+\rho_0, \epsilon_0-\rho_0])$ is at most $c\epsilon_0$. The same estimate holds in $\tilde{M}$, the universal cover of M. If ρ_0 is small enough – relative to ϵ_0 – the result follows. □

Remark 5.8. *There exists $k_0 > 0$, depending only on ρ_0, such that, if Ω is a convex subset in the disk of radius ρ_0 in H^2, the total curvature of the boundary of Ω is at most k_0.*

Proof. This follows from the Gauss-Bonnet Theorem applied to Ω, with k_0 equal to 2π plus the area of the hyperbolic disk of radius ρ_0. □

Proof of Lemma 5.4. If γ does not intersect the support of λ the statement obviously holds, so we suppose that some point $x \in \gamma$ is in the support of λ. Let D be the totally geodesic disk of radius $\epsilon_0/2$ centered at x and orthogonal, at x, to the support of λ. By construction D is disjoint from M_s.

Remark 5.8 shows that the total curvature of the connected component c of $D \cap S$ containing x is at most k_0. By Claim 5.7, each geodesic in the support of λ which intersects c makes with D an angle at least $\pi/4$. It follows that $i(c, \lambda) \leq 2k_0$. It also follows, since the length of γ is less than $\epsilon_0/4$, that γ can be deformed transversally to λ to a segment of c, so that $i(\gamma, \lambda) \leq i(c, \lambda)$. Therefore $i(\gamma, \lambda) \leq 2k_0$, and this proves the lemma. □

Proof of Proposition 5.3. Notice first that Lemma 5.4, although stated only for geodesic segments γ that are at distance at least $\epsilon_0/2$ from the cone singularities, actually applies without this hypothesis. This is because, by Lemma 2.7, the support of λ cannot enter the ϵ_0-neighborhood of the singular points, so that any part of γ at distance less than ϵ_0 from the singular set of S has zero intersection with λ.

Let $n \in \mathbb{N}$ be the unique integer such that $n\epsilon_0/4 \leq l_m(\gamma) < (n+1)\epsilon_0/4$. Then γ can be cut into a sequence of segments $\gamma_1, \cdots, \gamma_n$ of length $\epsilon_0/4$ and one last segment γ_{n+1} of length smaller than $\epsilon_0/4$. Lemma 5.4 can be applied to each of those segments, it yields that $i(\lambda, \gamma_i) \leq \lambda_1$, $1 \leq i \leq n+1$, so that

$$i(\lambda, \gamma) \leq (n+1)\lambda_1 \leq \left(\frac{4l_m(\gamma)}{\epsilon_0} + 1\right)\lambda_1 \ ,$$

this proves the proposition. $\square$

5.2 The grafted metric and the hyperbolic metric at infinity.

We consider here the relation between the grafted metric $G_\lambda(m)$ and the hyperbolic metric at infinity t.

One of the key properties of the "grafted metric" $G_\lambda(m)$" (see e.g. [KP94]) is that it is in the conformal class τ at infinity – more precisely, there is a natural "Gauss map" defined from the unit normal bundle of ∂M with its "grafted metric" to the boundary at infinity of M, which is conformal. This means that $G_\lambda(m)$ is conformal to t. Moreover, since the angles θ_i are in $(0, \pi)$, the intersection of the boundary of the convex core with the particles is at non-zero distance (for m) from the support of λ, so that the cone angles of the grafted metric at the intersections with the particles of the boundary of the convex core is equal to the cone angle of the corresponding singularities.

The fact t is conformal to $G_\lambda(m)$ translates as

$$t = e^{2u} G_\lambda(m) \ ,$$

where $u : \partial M \to \mathbb{R}$ is a function.

Lemma 5.9. *The function u is non-positive on ∂M.*

Proof. Consider two metrics g and g' with $g' = e^{2u}g$, and let K and K' be their curvatures. Then (see e.g. chapter 1 of [Bes87])

$$K' = e^{-2u}(\Delta u + K) \ .$$

We can apply this formula here with $g = G_\lambda(m)$ and $g' = t$, so that $K' = -1$ while $K \in [-1, 0]$. It takes the form:

$$\Delta u = -K - e^{2u} = |K| - e^{2u} \ ,$$

with $K \in [-1, 0]$ (this equation is understood in a distributional sense).

Since the cone angles are the same for t and for $G_\lambda(m)$, u is continuous and bounded at the singular points (see [Tro91]). Let $x_M \in S$ be a point where u achieves its maximum. Suppose first that x_M is not a singular point, then u is C^2 at x_M by elliptic regularity (see [Tro91]). Moreover $\Delta u \geq 0$ at x_M since x_M is a maximum of u. It follows that $e^{2u} \leq |K| \leq 1$, so that $u \leq 0$. To complete the proof it is sufficient to prove that u cannot achieve a positive maximum at a singular point of S. So we consider a singular point x_0 of S, and suppose that $u > 0$ at x_0. We will show that u cannot have a maximum at x_0.

Let D be the geodesic disk of radius r centered at x_0 in $(S, G_\lambda(m))$. Since λ does not enter a small neighborhood of x_0, D is hyperbolic, with only one cone singularity at x_0, if r is small enough. Let i_0 be the isometric map between D, with the metric $G_\lambda(m)$, and the hyperbolic disk H^2_α with one cone singularity of angle α, where α is the cone angle of S at x_0. Let $i_1 : D \to H^2_\alpha$ be the isometric embedding of (D, t) in H^2_α. Call v_0 the vertex of H^2_α, i.e., its singular point. Since $u > 0$ at x_0, if r is small enough, then

$$\forall x \in D \setminus \{x_0\}, d(i_1(x), v_0) > d(i_0(x), v_0) \ .$$

There is a natural complex map $\phi : H^2_\alpha \to H^2$, given in holomorphic coordinates centered at the singular point by $z \to z^{2\pi/\alpha}$. It is conformal and multiplies the metric by a factor $(2\pi/\alpha)^2 d(x, v_0)^{2(2\pi/\alpha-1)}$. Consider the composition

$$\Phi := \phi \circ i_1 \circ i_0^{-1} \circ \phi^{-1} : (\phi \circ i_0)(D) \to (\phi \circ i_1)^{-1}(D) .$$

It is a conformal map, with conformal factor equal to

$$(2\pi/\alpha)^2 d(i_1(x), v_0)^{2(2\pi/\alpha-1)} e^{2v} (2\pi/\alpha)^{-2} d(i_0(x), v_0)^{-2(2\pi/\alpha-1)} ,$$

with $v = u \circ (\phi \circ i_0)^{-1}$. This can be written as

$$\left(\frac{d(i_1(x), v_0)}{d(i_0(x), v_0)}\right)^{2(2\pi/\alpha-1)} e^{2v} ,$$

and is bigger than 1 since $u > 0$ and $d(i_1(x), v_0) > d(i_0(x), v_0)$.

Since Φ is a conformal map between two hyperbolic domains, its conformal factor cannot have a local maximum bigger than 1 at an interior point by the argument given at the beginning of this proof. Therefore, u cannot have a positive maximum at x_0. □

The following notion will be useful in this section and the next.

Definition 5.10. *A c-curve on ∂M is either a closed curve or a segment with endpoints at cone singularities, which does not contain any singular point (except at its endpoints if it's not a closed curve).*

We will sometimes implicitly consider c-curves up to homotopy in the complement of the singular points in ∂M. Each homotopy class (with fixed endpoints) contains a unique geodesic for any non-positively curved metric on ∂M (in particular for m, t and $G_\lambda(m)$). Given a c-curve γ, we will denote by $L_m(\gamma)$ (resp. $L_t(\gamma)$, $L_{G_\lambda(m)}(\gamma)$) the length of that geodesic for the corresponding metric.

Corollary 5.11. *Let γ be a c-curve in ∂M, then $L_t(\gamma) \leq L_{G_\lambda(m)}(\gamma)$.*

This follows directly from Lemma 5.9, since any minimizing c-curve in $(S, G_\lambda(m))$ has shorter length for t, and the minimizing curve in (S, t) in the same homotopy class is even shorter. Note also that for any c-curve γ, $i(\lambda, \gamma) \leq CL_m(\gamma)$. This follows from Proposition 5.3, and by the fact that the lengths of the c-curves which are segments between two singular points of S is bounded from below.

5.3 An upper bound on the lengths of the curves at infinity

The second step in the proof of Proposition 5.2 is a comparison between the lengths of c-curves in the metrics t and m.

Proposition 5.12. *There exists a constant $C > 0$ (independent of M) such that:*

1. *for each c-curve γ in ∂M, $L_t(\gamma) \leq CL_m(\gamma)$,*
2. *for each long tube T in the thin part of $(\partial M, m)$, T might also be a long tube for t, but its length for t is at most its length for m plus C.*

The proof uses some simple statements on the geometry of long hyperbolic tubes in (S, t). Recall (see [DP07]) that the Margulis Lemma applies to hyperbolic surfaces with cone singularities of angle at most θ, when $\theta \in (0, \pi)$: there exists a constant c_M, depending on θ only, such that the set of points where the injectivity radius is less than c_M is a disjoint union of cusps, disks centered at a cone singularity, and tubes with core of length less than $2c_M$.

We consider in this subsection a hyperbolic tube T, which can be described as isometric to the set of points at distance at most L (for some $L > 0$) from the unique simple closed geodesic in the quotient of the hyperbolic plane H^2 by a hyperbolic translation of length l. Moreover l is supposed to be small and L large, so that the lengths of the boundary components of T – which are both equal to $l\cosh(L)$ – are equal to c_M. We call σ the core of T, in other terms the unique simple closed geodesic contained in T, and we denote by σ_M the c_M-neighborhood of σ – the set of points at distance at most c_M from σ in T.

Lemma 5.13. *There exists a constant $C > 0$ such that $i(\lambda, \partial\sigma_M) \leq Ce^{-L}$: the intersection of λ with the boundary of σ_M is at most Ce^{-L}.*

Proof. Any maximal embedded geodesic segment in T intersects exactly once σ, but also each of the two connected components of $\partial\sigma_M$. It follows that the intersection with λ of each of the connected components of $\partial\sigma_M$ is equal to $i(\lambda, \sigma)$. But since the length of σ is $l = e^{-L}$, Proposition 5.3 – applied to long segments that wrap many times around σ – shows that $i(\lambda, \sigma) \leq Ce^{-L}$. $\square$

Lemma 5.14. *There exists a constant $C > 0$ such that, if g is an embedded maximal geodesic segment in T, then the length of the orthogonal projection on σ_M of $g \cap (T \setminus \sigma_M)$ is at most C.*

Proof. If $g \subset \sigma$, then $g \cap (T \setminus \sigma_M) = \emptyset$ and the result applies. We suppose from here on that g is not contained in σ. If g is contained in one connected component of the complement of σ in T then, since g is embedded, its orthogonal projection on σ is injective, so that the length of its orthogonal projection is bounded by the length of σ. Otherwise, it follows from standard hyperbolic geometry arguments that g intersects σ exactly once.

Consider the universal cover $\tilde{T}$ of T, it is isometric to the set of points at distance at most L from a geodesic $\tilde{\sigma} \subset H^2$ which is the lift of σ. Choose one of the connected components, say $\tilde{g}$, of the lift of g to $\tilde{T}$. It intersects the lift of $\partial\sigma_M$ with an angle which is bounded from below – otherwise g could not intersect σ. It follows from this, and from elementary geometric properties of the hyperbolic plane, that the length of the orthogonal projection on $\tilde{\sigma}$ of each of the segment of $\tilde{g}$ outside the lift of σ_M is bounded from above by a constant. $\square$

Corollary 5.15. *There exists a constant $C > 0$ such that, whenever g_0 is a maximal geodesic segment in T such that the orthogonal projection of g_0 on σ is injective, then $i(\lambda_{|T\setminus\sigma_M}, g_0) \leq C$.*

Proof. Let c be a maximal geodesic segment in the intersection with T of the support of λ, and let c' be one of the connected components of $c \cap T \setminus \sigma_M$. Since both c and g_0 are geodesic segments, the union of the orthogonal projections on σ of the segments of g_0 and of c between two successive intersections between them covers σ.

It follows that the number of intersections between c' and g_0 is at most equal to $(l_{c'} + l_{g_0})/l$, where $l_{c'}$ is the length of the orthogonal projection of c' on σ and l_{g_0} is the length of the orthogonal projection of g_0 on σ (and l is the length of σ).

But the hypothesis on g_0 shows that $l_{g_0} \leq l$, while Lemma 5.14 shows that $l_{c'} \leq C$. So the number of intersections between c' and g_0 is at most Ce^L, where C is some positive constant.

Since this inequality applies to all geodesic segments in the support of λ, we find that

$$i(\lambda_{T\setminus\sigma_M}, g_0) \leq Ce^L i(\lambda, \partial\sigma_M) ,$$

and Lemma 5.13 then shows that $i(\lambda_{T\setminus\sigma_M}, g_0)$ is bounded by a positive constant. $\square$

Proof of Proposition 5.12. Let γ be a c-curve in ∂M. It follows from Proposition 5.3 that

$$L_{G_\lambda(m)}(\gamma) \leq L_m(\gamma) + i(\lambda, \gamma) \leq C(L_m(\gamma) + 1) ,$$

where here again C is a constant depending only on the topology of M. Moreover Corollary 5.11 indicates that

$$L_t(\gamma) \leq L_{G_\lambda(m)}(\gamma) ,$$

and point (1.) follows.

For point (2.) consider a closed geodesic γ contained in the union of T and of the thick part of ∂M, such that

- the intersection of γ with T has two connected components γ_1 and γ_2,
- the intersection of γ with the thick part of ∂M (for m) has two connected components γ_1' and γ_2', and each has length bounded by C.

If T separates the boundary component of ∂M containing it, γ has to go through T twice, otherwise it is not necessary but it is still possible to choose γ with this property, and both cases can then be treated in a uniform manner.

Once such a curve γ has been found, it is possible to change it by Dehn twists so that, in addition to the conditions above, the segments γ_1 and γ_2 "wrap" at most once around T, i.e., their orthogonal projection to the core σ of T is injective. This is achieved by "untwisting" γ as much as is necessary.

Denote as above by σ the core of T, and by σ_M the set of points at distance at most c_M from σ. Since γ wraps at most once around T, the length of the intersection of γ_1 and γ_2 with σ_M is at most $3c_M$. It then follows from Proposition 5.3 that

$$i(\lambda_{|\sigma_M}, \gamma) \leq C \ ,$$

where C is some positive constant. By the same proposition, the intersection of γ with the restriction of λ to the thick part of $(\partial M, m)$ is at most C. But Corollary 5.15 shows that

$$i(\lambda_{|T \setminus \sigma_M}, \gamma) \leq C \ .$$

Putting together those estimates we obtain that $i(\lambda, \gamma) \leq C$, where C is yet another positive constant. The definition of the grafted metric then proves that

$$L_{G_\lambda(m)}(\gamma) \leq L_m(\gamma) + C \ .$$

Finally Lemma 5.9 indicates that the length of γ for t is less than that for $G_\lambda(m)$. The result follows. $\square$

5.4 A bound on the length of the earthquake lamination

We now switch from 3-dimensional to 2-dimensional geometry to show that an upper bound on the length of curves in $(\partial M, t)$ – relative to the length of the same curves in $(\partial M, m)$, as stated in Proposition 5.12 – implies a lower bound on the same lengths. Proposition 5.2 will follow.

We consider a closed surface Σ, with some marked points $x_1, \cdots, x_{n_0}$, and an angle $\theta_i \in (0, \pi)$ attached to x_i.

Proposition 5.16. *For each $C > 0$ there is a constant $C' > 0$ as follows. Let $h, h' \in \mathcal{H}_{\Sigma,x,\theta}$ be two hyperbolic metrics such that:*

1. *for each c-curve γ in Σ, $L_{h'}(\gamma) \leq C L_h(\gamma)$,*
2. *for each long tube T in the thin part of (Σ, h), T might also be a long tube for h', but its length for h' is at most its length for h plus C.*

Let $\nu \in \mathcal{M}_{\Sigma,x}$ be the measured lamination such that $h' = E_r(\nu)(h)$. Then the length $L_h(\nu)$ is at most C'.

The proof of Proposition 5.16 will use a basic estimate on the variation of the length of curves under an earthquake, essentially taken from [BS06].

Proposition 5.17. *Let $m \in \mathcal{ML}_{\Sigma,x}$ be a measured lamination, let $g \in \mathcal{H}_{\Sigma,x,\theta}$ be a hyperbolic metric with cone singularities, and let $g' := E^r_m(g)$. Let γ be a c-curve. Then*

$$|L_g(\gamma) - L_{g'}(\gamma)| \leq i(m, \gamma) \leq L_g(\gamma) + L_{g'}(\gamma) \ .$$

Proof. The upper bound on $i(m, \gamma)$ can be found in [BS06] (Lemma 7.1, p. 76); it is stated there for closed curves, but the proof extends directly to segments between two singular points.

For the lower bound on $i(m, \gamma)$, suppose first that the support of m is a disjoint union of simple closed curves. Consider the geodesic (for g) γ_0 homotopic to γ in Σ_x, and let γ_1 be its image by the earthquake E^r_m, along with the union of the segments in the support of m between two points corresponding – after the earthquake – to one intersection of m with γ_0. γ_1 is homotopic to γ_0 in Σ_x. Clearly $L_{g'}(\gamma_1) = L_g(\gamma) + i(m, \gamma)$, while $L_{g'}(\gamma) \leq L_{g'}(\gamma_1)$. It follows that $L_{g'}(\gamma) \leq L_g(\gamma) + i(m, \gamma)$. The same inequality also holds with g and g' exchanged, and the lower bound on $i(m, \gamma)$ follows. The result when m is a general lamination – not rational – holds by density of the rational laminations in $\mathcal{ML}_{\Sigma,x}$. $\square$

We now return to the notations used in Proposition 5.16. Note that the support of ν is a geodesic lamination in (Σ, g). It is therefore possible to consider the intersection of ν with the thin (resp. thick) part of Σ for g, which we call ν_t (resp. ν_T). The same decomposition can be done for g', leading to ν'_t and ν'_T.

We first state a basic property of hyperbolic surfaces, which is necessary below.

Lemma 5.18. *There exist $r > 0$, $C > 0$ and $\theta_0 \in (0, \pi)$, depending only on the supremum θ_M of the θ_i and on the genus of Σ, such that, for any $x \in \Sigma_T$ and any geodesic segment γ_0 of length $2r$ centered at x, there exists a closed geodesic in Σ of length at most C intersecting γ_0 with angle at least θ_0.*

Proof. Note that any maximal segment in the thick part of a topologically non-trivial hyperbolic surface (with cone singularities of angle less than π) intersects some closed geodesic, of length bounded by a constant C.

The statement therefore follows from a straightforward compactness argument. Indeed, if the constant $\theta < \theta_M$ did not exist, there would be a sequence of thick hyperbolic surfaces with boundary $\Sigma_{T,n}$ (with cone singularities of angles less than π), for which the optimal value of r would go to infinity, or the optimal value of θ would go to π, as $n \to \infty$. This sequence could be taken of fixed topology, and the diameter of those surfaces would then be bounded, so that r would necessarily be bounded.

We could then choose a converging subsequence, and obtain a thick hyperbolic surface (with cone singularities of angle less than θ_M) for which some maximal geodesic segment intersects no closed geodesic of length less than C transversally, a contradiction. □

Lemma 5.19. *There exists a constant C (depending only on the genus of Σ) so that the length of ν_T is at most C.*

Proof. Let $r_0 > 0$ be smaller than the injectivity radius of (Σ, h) at each point of Σ_T. There exists another number $r_1 \in (0, r_0)$ with the following property: if γ_0 and γ_1 are two disjoints geodesics in H^2 and $x \in \gamma_0$ is at distance at most r_1 from γ_1, then any geodesic intersecting γ_0 at distance less than r_1 from x and making an angle bigger than θ_0 with γ_0 intersects γ_1 at distance at most r_0 from x.

Choose a large constant $C_1 > 0$. If the length of ν_T were bigger than some large constant, the sum of the weights of the segments of the support of ν intersecting some geodesic disk of radius r_1 and center $x \in \text{supp}(\nu) \cap \Sigma_T$ would be bigger than C_1. Applying the previous lemma, with γ_0 equal to a segment containing x in the support of ν, would yield a closed curve c in Σ_T, of bounded length, such that $i(c, \nu)$ is arbitrarily large.

Proposition 5.17 would then show that the length of c for h' is much larger than the length of c for h, contradicting point (1) in the hypothesis of Proposition 5.16. □

Lemma 5.20. *There exists a constant $C > 0$ as follows. Let $\gamma \subset \Sigma_T$ be a geodesic segment of length at most c_M. Then*

$$i(\nu, \gamma) \leq CL(\nu_T) \ .$$

Proof. We will consider the case when ν is rational, the general case follows by density of the rational measured laminations.

Let r be the injectivity radius of Σ_T. Let ν_γ be the union of the intersections with Σ_T of all geodesic segments centered at a point $x \in \gamma$, of length $2r$, in the support of ν. Each of those segments has length at least r, since at least one side of x is contained in Σ_T.

By definition of r, those segments intersect γ exactly once. Moreover, the length of ν_T is larger than the sum over the segments of their length times their weight (this sum is finite since ν is rational). But this sum is at least $ri(\nu, \gamma)$, so that $L(\nu_T) \geq ri(\nu, \gamma)$. This proves the lemma. □

Lemma 5.21. *There exists a constant $C > 0$ such that, if T is a tube of length $2L$ in (Σ_t, h), with core σ, then $i(\nu, \sigma) \leq Ce^{-L}$.*

Proof. $L_h(\sigma) = c_1 e^{-L}$, where c_1 is some constant. Point (1) in the hypothesis of Proposition 5.16 shows that the length of σ for h' is at most $c_2 e^{-L}$, where c_2 is another positive constant. But Proposition 5.17 then yields the result. □

In the next lemma, we call σ_M the set of points at distance at most c_M from σ.

Lemma 5.22. *There exists a constant $C > 0$ such that, if T is a tube of length $2L$ in (Σ_t, h), with core σ, then the length for h of the restriction of ν to σ_M is at most C.*

Note that there is no reason to believe that this proposition is optimal; indeed, it appears quite reasonable to think that the bound could be improved to Ce^{-L}. The bound given here, however, is both sufficient for our needs and easier to obtain.

Proof. We know by Lemma 5.21 that $i(\nu, \sigma) \leq C_1 e^{-L}$, where $C_1 > 0$ is some constant. It follows that, if $L_h(\nu_{|\sigma_M})$ is larger than some constant C_2, then each leaf of ν intersects σ with a very small angle, and for any geodesic segment γ going through T and intersecting σ with angle bigger than $\pi/4$, $i(\gamma, \nu_{|\sigma_M}) \geq C_2 e^L$.

Let γ_1 be a closed geodesic in (Σ, h) which has two segments in Σ_T and two segments going through T. Furthermore we choose γ_1 with the smallest length. Since Σ_T has bounded diameter, there is $C > 0$ such that $\ell_h(\gamma_1) \leq C + 2T$. Since γ_1 has minimal length, it intersects σ with angle at most $\pi/4$ at each of the two intersections. Then $i(\nu, \gamma_1) \geq 2C_2 e^L$, so that, by Proposition 5.17, $L_{h'}(\gamma_1)$ is much larger than $L_h(\gamma_1)$. This contradicts the hypothesis of Proposition 5.16. ∎

Proof of Proposition 5.16. According to Lemma 5.19, the length of the restriction of ν to the thick part of Σ is bounded by a constant (depending only on the genus of Σ). Σ is the union of Σ_T and a finite set of long thin tubes, the number of those tubes being at most $3g - 3$, where g is the genus of Σ. Let T be one of those tubes, and let σ be its core. Then $L(\nu_{|\sigma_M})$ is bounded by a constant by Lemma 5.22. Moreover, the length of each maximal segment of the support of ν in $T \setminus \sigma_M$ is at most $2e^L$, and each is contained in a maximal segment in T (contained in the support of ν) which intersects σ once. Since $i(\nu, \sigma) \leq Ce^L$, the length of the restriction of ν to $T \setminus \sigma_M$ is at most $4C$. Summing all contributions to the length of ν yields the desired result. ∎

Proof of Proposition 5.2. The statement clearly follows from Proposition 5.12 and from Proposition 5.16. ∎

6 The conformal structure at infinity

This section contains the proof of Theorems 1.7 and 1.14, mostly as a consequence of Lemma 5.2.

6.1 A topological lemma

We first state a simple topological lemma, necessary below to apply Proposition 5.2 as directly as possible. We fix a closed surface S of genus at least 2, a n-tuple of points $x = (x_1, \cdots, x_{n_0})$ and a n_0-tuple of angles $\theta = (\theta_1, \cdots, \theta_{n_0}) \in (0, \pi)^n$.

Lemma 6.1. *Let $c > 0$, and let $K \subset \mathcal{H}_{S,x,\theta}$ be a compact subset. The set of all elements of $\mathcal{H}_{S,x,\theta}$ obtained by a right earthquake along a measured lamination of length at most c on an element of K is relatively compact.*

Proof. Let $m \in K$. The set of measured laminations $l \in \mathcal{ML}$ of length less than c for m is compact in $\mathcal{ML}_{S,x}$. Since the earthquake map is continuous relative to the measured lamination factor, the set

$$E_r(\{l \in \mathcal{ML}_{S,x} \mid L_m(l) \leq c\} \times \{m\})$$

is compact in $\mathcal{H}_{S,x,\theta}$.

Again because the earthquake map E_r is continuous, it follows that there is a neighborhood U_m of m in $\mathcal{H}_{S,x,\theta}$ such that the image by E_r of

$$\{(l, m') \in \mathcal{ML}_{S,x} \times U_m \mid L_{m'}(l) \leq C\}$$

is relatively compact.

Since K is compact, it is covered by finitely many such neighborhoods U_{m_i}, for m_i in K. The result follows. ∎

6.2 Compactness relative to the conformal structure at infinity

The previous considerations lead to a simple proof of Proposition 1.6 from Proposition 5.2 and Lemma 3.15.

Consider a sequence $(g_n)_{n\in\mathbb{N}}$ of quasifuchsian metrics with particles, as in Proposition 1.6. Let $(m_n)_{n\in\mathbb{N}}$ be the sequence of induced metrics on the boundary of the convex core, and let t_n be the sequence of hyperbolic metrics in the conformal class at infinity τ_n. Since $(\tau_n)_{n\in\mathbb{N}}$ converges by the hypothesis of Proposition 1.6, $(t_n)_{n\in\mathbb{N}}$ converges to a limit t, so it remains in a compact subset of $\mathcal{H}_{S,x,\theta}\times\mathcal{H}_{S,x,\theta}$.

But m_n is obtained from t_n by an earthquake along a measured lamination ν_n which, by Proposition 5.2, has bounded length. Lemma 6.1 therefore shows that $(m_n)_{n\in\mathbb{N}}$ remains in a compact subset of $\mathcal{H}_{S,x,\theta}\times\mathcal{H}_{S,x,\theta}$. We can therefore extract a sub-sequence so that $(m_n)_{n\in\mathbb{N}}$ converges.

Lemma 3.15 then shows that $(g_n)_{n\in\mathbb{N}}$ has a subsequence which converges to a quasifuchsian metric with particles. This proves the proposition.

6.3 Proof of Theorem 1.7

We are now ready to prove Theorem 1.7. It is helpful to introduce additional notations:

- $\mathcal{M}_{S,x} := \cup_{\theta\in(0,\pi)^N}\mathcal{M}_{S,x,\theta}$ is the space of quasifuchsian metrics with particles on $S\times\mathbb{R}$ with a fixed number of particles but with varying angles,
- $\mathcal{H}_{S,x} := \cup_{\theta\in(0,\pi)^N}\mathcal{H}_{S,x,\theta}$ is the space of hyperbolic metrics on S with a fixed number of cone singularities but with varying angles,
- $\Delta_{S,x} := \cup_{\theta\in(0,\pi)^N}\mathcal{H}_{S,x,\theta}\times\mathcal{H}_{S,x,\theta}$ is a kind of diagonal with respect to the angle variable in $\mathcal{H}_{S,x}\times\mathcal{H}_{S,x}$.

Note that, by a result of Troyanov [Tro91] already mentioned above, $\mathcal{H}_{S,x}$ can be naturally identified with $\mathcal{T}_{S,x}\times(0,\pi)^N$. The notation is nonetheless useful in the argument presented here.

Consider the natural map:

$$\Phi_{S,x} : \mathcal{M}_{S,x}\to\mathcal{H}_{S,x}\times\mathcal{H}_{S,x}$$

sending a hyperbolic metric with particles on $S\times\mathbb{R}$ (with cone angles given by the θ_i) to the conformal structures at $\pm\infty$. It follows from the definition that the image of $\Phi_{S,x}$ is contained in $\Delta_{S,x}$.

Let $\Phi_{S,x,\theta}$ be the restriction of $\Phi_{S,x}$ to $\mathcal{M}_{S,x,\theta}$, for a fixed $\theta\in(0,\pi)^N$. The main result of [MS06] is that – in a slightly more general context, allowing for more topology – $\Phi_{S,x,\theta}$ is a local homeomorphism from $\mathcal{M}_{S,x,\theta}$ to $\mathcal{H}_{S,x,\theta}\times\mathcal{H}_{S,x,\theta}$. It follows that $\Phi_{S,x}$ is a local homeomorphism from $\mathcal{M}_{S,x}$ to $\Delta_{S,x}$. Moreover, $\Phi_{S,x}$ is proper by Proposition 1.6, so that it is a covering of $\mathcal{T}_{S,x,\theta}\times\mathcal{T}_{S,x,\theta}$.

To prove that $\Phi_{S,x}$ is a (global) homeomorphism we need to show that some elements of the target space have exactly one inverse image. Suppose that for all $i\in\{1,\cdots,N\}$, $\theta_i=2\pi/k_i$, where $k_i\in\mathbb{N}, k_i\geq 2$. Let $\tau_+,\tau_-\in\mathcal{H}_{S,x,\theta}$. There exists a finite covering $\pi:\overline{S}\to S$, with ramification of order k_i at the x_i, such that τ_+ (resp. τ_-) lifts to a non-singular hyperbolic metric $\overline{\tau}_+$ (resp. $\overline{\tau}_-$). By the Bers double uniformization theorem [Ber60] $\overline{\tau}_+$ and $\overline{\tau}_-$ are in the conformal class at infinity of a unique quasifuchsian hyperbolic metric, say $\overline{g}$, on $\overline{S}\times\mathbb{R}$. Since it is unique, $\overline{g}$ is invariant under the deck transformations of π, so that $\overline{g}$ is the pull-back to $\overline{S}\times\mathbb{R}$ of a hyperbolic metric g on $S\times\mathbb{R}$, with cone singularities of angle θ_i along $\{x_i\}\times\mathbb{R}$, $1\leq i\leq N$. This construction also shows that g is unique – since any other hyperbolic metric with particles of the given angles would lift to a non-singular quasifuchsian metric on $\overline{S}\times\mathbb{R}$, which would have to be $\overline{g}$. This shows that (τ_+,τ_-) has a unique inverse image by $\Phi_{S,x}$, so that $\Phi_{S,x}$ is a homeomorphism from $\mathcal{M}_{S,x}$ to $\Delta_{S,x}$.

6.4 Proof of Theorem 1.14

We need another natural map.

Definition 6.2. *Let $\Psi_{S,x,\theta}:\mathcal{H}_{S,x,\theta}\times\mathcal{H}_{S,x,\theta}\to\mathcal{H}_{S,x,\theta}\times\mathcal{H}_{S,x,\theta}$ be defined as follows. Given $(t_+,t_-)\in\mathcal{H}_{S,x,\theta}\times\mathcal{H}_{S,x,\theta}$ and $\theta=(\theta_1,\cdots,\theta_{n_0})\in(0,\pi)^n$, there is by Theorem 1.7 a unique quasifuchsian metric with particles $g\in\mathcal{M}_{S,x,\theta}$ such that $\Phi_{S,x,\theta}(g)=([t_+],[t_-])$. Then*

$$\Psi_{S,x,\theta}(t_+,t_-)=(m_+,m_-)\ ,$$

where m_+ and m_- are the conformal classes of the induced metrics on the upper and lower boundary components of the convex core of $(S \times \mathbb{R}, g)$, respectively.

According to Proposition 5.2 and Lemma 6.1, if $(m_+, m_-) = \Psi_{S,x,\theta}(t_+, t_-)$, then m_+ is C_q-quasi-conformal to t_+, and m_- is C_q-quasi-conformal to t_-. This shows that $\Psi_{S,x,\theta}$ is proper and extends continuously to a map which is the identity on the boundary at infinity of $\mathcal{H}_{S,x,\theta} \times \mathcal{H}_{S,x,\theta}$, so that it is onto. This proves Theorem 1.14.

7 Some questions and remarks

7.1 Some questions.

The quasifuchsian cone-manifolds described above are direct extensions of the "usual" quasifuchsian hyperbolic manifolds which have received much attention over the last couple of decades. It is quite natural to wonder whether some properties which have been conjectured in the smooth case can be extended to the singular setting.

Question 7.1. *Does uniqueness hold in Theorem 1.14?*

Another natural question, which is "dual" to the previous one in a precise sense (see e.g. [Sch03]) concerns the measured bending lamination on the boundary of the convex core.

Question 7.2. *Does uniqueness hold in Theorem 1.13?*

The same questions can be asked for submanifolds of quasifuchsian cone-manifolds which are convex but have a smooth boundary, which is orthogonal to the singular locus. In the smooth case it is known [Sch06] that one can prescribe the induced metric on the boundary, as well as the its third fundamental form (the smooth analog of the measured bending lamination) and that each is obtained uniquely, it would be interesting to know whether the same is true for quasi-fuchsian cone-manifold. The methods of [Sch06] do not appear to extend directly to the singular case.

Note also that those questions are not necessarily restricted to quasifuchsian cone-manifolds, and could also be asked for "convex co-compact cone-manifolds", if that term is understood in a proper way.

7.2 AdS manifolds with particles.

Mess [Mes07, ABB$^+$07] discovered a remarkable analogy between quasifuchsian hyperbolic 3-manifolds and globally hyperbolic maximal compact (GHMC) AdS manifolds. In particular he proved an analog of the Bers double uniformization theorem form GHMC AdS manifolds: on a manifold homeomorphic to $S \times \mathbb{R}$, where S is a closed surface of genus at least 2, the space of GHMC AdS manifolds is parametrized by the product of two copies of the Teichmüller space of S, through the "left" and "right" parts of the holonomy representation.

GHMC AdS manifolds also have a convex core, whose boundary has a hyperbolic induced metric, as in the quasifuchsian setting, and is pleated along a measured lamination. The analog of Theorem 1.13 holds in that context [BS09]: any two measured laminations that fill S can be obtained as the bending lamination of the boundary of the convex core. But the uniqueness remains elusive, as in the quasifuchsian setting. Moreover, the analog of Theorem 1.14 is also conjectured to be true but no result is known.

It is also possible to consider GHMC AdS manifolds with "particles", i.e., cone singularities along time-like geodesics, for which the angle is less than π. The analog of the Bers double uniformization theorem (more directly, the analog of Theorem 1.7) holds in this AdS setting [BS06]. The analog of Theorem 1.13 is also true in that setting [BS09]. However no analog of Theorem 1.14 is known.

7.3 The renormalized volume.

Theorem 1.7 for quasifuchsian manifolds with particles has possible applications to the Teichmüller theory of hyperbolic surfaces with cone singularities (of fixed angles) on a surface. Indeed it was remarked in [KS08a] that

the definition of the renormalized volume of a quasifuchsian 3-manifolds extends to manifolds with particles. Knowing Theorem 1.7, it is possible to remark that the key property of the renormalized volume – to be a Kähler potential for the Weil-Petersson metric on Teichmüller space – extends to the natural Weil-Petersson metric on the Teichmüller space of hyperbolic metrics with cone singularities (of prescribed angle less than π) on a surface; the proof from [KS08a, KS09] directly extends to this setting.

One direct consequence is that this Weil-Petersson metric is Kähler, as was discovered by Schumacher and Trapani [ST08] by other means. This metric, however, seems to depend on the choice of the cone angles.

Another possible application is to some properties of the grafting map considered on hyperbolic surfaces with cone singularities of angle less than π, as considered in [KS08b, KS09]. This is however less directly related to what we are doing here, since it only uses the geometry of 3-dimensional hyperbolic ends – rather than quasifuchsian metrics – with particles.

A Quasiconformal estimates

This appendix contains the proof of Proposition 1.15. The first step is a simple extension to hyperbolic surfaces with cone singularities of some classical tools concerning pants decompositions.

A.1 Pants decompositions

The content of this subsection is probably well known, see e.g. [DP07] for closely related considerations. We include this material for completeness.

Let S be a closed surface, and let h be a hyperbolic metric on S with cone singularities at some points $x_1, \cdots, x_{n_0}$, with cone angles $\theta_1, \cdots, \theta_{n_0} \in (0, \pi)$. If h had cusps – or geodesic boundary components – at the x_i rather than cone singularities, it would be quite natural to consider pants decompositions of (S, h). With cone singularities of angles less than π, it remains possible.

Definition A.1. *A singular pair of pants is a hyperbolic surface with geodesic boundary, possibly containing cone singularities of angle less than π, which is either:*

- *a hyperbolic pair of pants (with geodesic boundary) containing no cone singularity,*
- *a hyperbolic annulus with geodesic boundary containing exactly one cone singularity,*
- *a hyperbolic disk with geodesic boundary containing exactly two cone singularities.*

Given a singular hyperbolic pair of pants, its three geodesic boundary components or cone singularities will be called its *legs*. We hope that the reader will excuse us for this weird and perhaps confusing terminology.

Definition A.2. *A* **pants decomposition** *of S is a decomposition $S = S_1 \cup \cdots \cup S_n$ of S as the union of finitely many subsurfaces with disjoint interior, each of which is a singular pair of pants.*

It is implicit in this definition that the boundary of the S_i contains no cone singularities; the cone singularities are each contained in the interior of one of the singular pairs of pants.

Lemma A.3. *There exists a constant $C_p > 0$ such that, for any choice of S and h, (S, h) has a pants decomposition with all boundary curves of length less than C_p.*

Sketch of the proof. A standard recursive argument (see e.g. [BP92]) reduces the proof to showing that, for any hyperbolic surface with cone singularities (of angle less than π) and geodesic boundary, there is a simple closed geodesic of length at most C_p which is not homotopic to a singular point or to a boundary component.

This in turn follows from other standard arguments, for instance based on comparing the area of the surface (given by a suitable Gauss-Bonnet formula, see e.g. [Tro91]) to the area of embedded geodesic disks. □

Definition A.4. *Let P be a singular pant. Its leg invariants are the length of its geodesic boundary components and the angles at its cone singularities.*

For instance, the boundary invariants of a (non-singular) hyperbolic pair of pants are the lengths of its boundary components.

Lemma A.5. *Each hyperbolic pair of pants is uniquely determined, up to isometry, by its leg invariants and by the type of its "legs" – whether they are boundary components or cone singularities.*

The proof follows the classical arguments used for non-singular hyperbolic pairs of pants, it is based on elementary properties of some hyperideal hyperbolic triangles stated below in three propositions (the first two have probably been known since Lobachevsky).

Recall that a hyperideal triangle can be defined, using the projective model of the hyperbolic plane, as a triangle which might have its vertices either in the hyperbolic plane, on its ideal boundary, or outside the closure of the hyperbolic plane (considered as the interior of a disk in the projective plane), but with all edges intersecting the hyperbolic plane. A vertex is then ideal if it is on the ideal boundary, and strictly hyperideal if it is outside the closed disk.

Recall also that given a point v_0 outside the closure of the projective model of H^2 (in the projective plane), there is a unique hyperbolic geodesic, v_0^*, such that any the intersection with the projective model of H^2 of any projective line containing v_0 is orthogonal to v_0^*. This geodesic is called the line *dual* to v_0.

We introduce here a slightly restricted notion of hyperideal triangle.

Definition A.6. *An* **extended hyperbolic triangle** *is a hyperbolic triangle with one or more strictly hyperideal vertices and its other vertices in the "interior" of the hyperbolic plane. A* **truncated hyperbolic triangle** *is the intersection of an extended hyperbolic triangle with the hyperbolic half-planes bounded by the lines dual to its strictly hyperideal vertices (and not containing the endpoints of the edges going towards those vertices).*

For instance, a right-angle hyperbolic hexagon can be considered – in two ways – as a truncated hyperbolic triangle, with three strictly hyperideal vertices. Given a hyperbolic triangle, its *angles* are the hyperbolic angles at the non-hyperideal vertices and the length of its intersections with the lines dual to the strictly hyperideal vertices. Notes that those lengths can quite naturally be considered as angles (they are then imaginary numbers) but it is not necessary to enter such considerations here (see e.g. [Sch98, Sch01] for more details).

There is a natural way to define the *edge lengths* of an extended hyperbolic triangle. The length of the edge joining two vertices v and v' is:

- the hyperbolic distance between v and v', if neither v nor v' is strictly hyperideal,
- the hyperbolic distance between v and the line dual to v', when v' is hyperideal but v is not,
- the distance between the lines dual to v and v', when both are strictly hyperideal.

It is useful to remark that the lengths and angles of an extended hyperbolic triangle satisfy a natural extension of the cosine formula. Moreover it is quite easy to check that an extended hyperbolic triangle, with vertices of given type, is uniquely determined by two lengths and one angle, or by two angles and one length.

Lemma A.7. *An extended hyperbolic triangle is uniquely determined by the type of its vertices – whether they are "usual" or strictly hyperideal vertices – and its edge lengths.*

Proof. This statement is classical for "usual" hyperbolic triangles, with no strictly hyperideal vertex. It is also well-known for triangles with three strictly hyperideal vertices, see [BP92].

Consider an extended hyperbolic triangle, with exactly one strictly hyperideal vertex, say v_1, and two usual vertices, v_2 and v_3. Let l_i be the length of the edge between v_j and v_k, for $\{i, j, k\} = \{1, 2, 3\}$. Consider l_2, l_3 as fixed, then l_1 is easily seen (using for instance the cosine formula for extended hyperbolic triangles) to be a strictly increasing function of the angle δ at v_1. This proves the lemma in this case.

Consider now the situation where v_1 is a "usual" vertex, while v_2 and v_3 are strictly hyperideal. Then given v_1, the positions of the lines dual to v_2 and v_3 are completely determined by the angle α at v_1. Moreover the distance between those lines, which by definition is equal to l_1, is a strictly increasing function of α. This shows the result in this last case. $\square$

The same arguments can be used to prove the "dual" lemma, concerning the possible angles. Here we need a more precise statement, in addition to the fact that the angles determine the triangle we need to know that a large class of triples of angles can actually be realized.

Lemma A.8. *An extended hyperbolic triangle is uniquely determined by the type of its vertices – whether they are "usual" or strictly hyperideal vertices – and its angles. The angles at the "usual" vertices can take any value in $(0, \pi/2)$, while the "angles" at the strictly hyperideal vertices can be any numbers in $(0, \infty)$.*

Proof. Again the case where all vertices are "usual" is classical, while the case where all three vertices are strictly hyperideal is well-known.

Consider a triangle T with one "usual" vertex, say v_1, and two strictly hyperideal vertices, v_2 and v_3. Let e_1, e_2, e_3 be the edges opposite to v_1, v_2, v_3 respectively. The triangle T is completely determined by the length l_1 of the edge e_1 and by the "angles" α_2 and α_3, that is, the lengths of the segments of the lines v_2^*, v_3^* dual to v_2, v_3 between their intersections with e_1 and with e_3 and e_2, respectively.

Given α_2 and α_3, the possible values of l_1 vary between a minimal value $l_{1,m}$ and a maximal value $l_{1,M}$. Suppose for instance that $\alpha_2 \geq \alpha_3$, then $l_{1,m}$ corresponds to the situation where e_3 is reduced to a point. Then, after truncation, T corresponds to a quadrilateral with 3 right angles. The last angle, between v_2^* and e_2, has to be less than $\pi/2$ by the Gauss-Bonnet theorem. This means that for l_1 slightly larger than $l_{1,m}$, $\alpha_3 > \pi/2$. On the other hand, $\alpha_3 \to 0$ as $l_1 \to l_{1,M}$, and α_3 is a decreasing function of $l_1 \in (l_{1,m}, l_{1,M})$. This proves the lemma for triangles with two strictly hyperideal vertices.

Consider now a triangle T' with one strictly hyperideal vertex, say v_1, and two "usual" vertices, v_2 and v_3. Consider $\alpha_2, \alpha_3 \in (0, \pi/2)$ as fixed, T' is then entirely determined by l_1. l_1 can vary between a minimal value $l_{1,m} > 0$ and ∞, where $l_{1,m}$ corresponds to the case where v_1 is an ideal vertex. The angle α_1 then varies between 0 and ∞, and is a strictly increasing function of l_1. The result follows. $\square$

Lemma A.9. *Each singular pair of pants has a unique decomposition as the union of two copies of a truncated hyperbolic triangle (glued along their common boundary).*

Proof. Let v_1, v_2, v_3 be the three legs – which could be either singular points or boundary components. There is a unique homotopy class of embedded segment joining v_i to v_j, for $i \neq j$. Those three segments can be uniquely realized as minimizing geodesics, which are then orthogonal to the boundary components. Cutting the pair of pants along those three geodesic segments yields two extended hyperbolic triangles, glued along their edges. Those two extended triangles have the same edge lengths, so that they are isometric by Lemma A.7. $\square$

Proof of Lemma A.5. By Lemma A.8, the two extended triangles glued to obtain a hyperbolic pair of pants are uniquely determined by their angles, which can take any value as long as the angles at the "usual" vertices are less than $\pi/2$. This shows that hyperbolic pairs of pants are uniquely determined by their leg invariants, and any values are possible as long as the angles at the singular points are less than π. $\square$

We now turn to the parameterization of hyperbolic metrics with cone singularities by Fenchel-Nielsen type coordinates. We first state a lemma on the existence and uniqueness of a pants decomposition from topological data, leaving the proof to the reader since it is the same as in the non-singular case.

Lemma A.10. *A pants decomposition is uniquely determined by the choice of the boundary curves $\gamma_1, \cdots, \gamma_N$, considered as simple closed curves in $S \setminus \{x_1, \cdots, x_{n_0}\}$, under the hypothesis that:*

- *the γ_i can be realized as pairwise disjoint curves,*
- *each connected component of their complement is either a pair of pants containing none of the x_i, or a cylinder containing exactly one of the x_i, or a disk containing exactly two of the x_i.*

Finally we state the main consequence, on the parameterization of hyperbolic metrics with cone singularities of fixed angle by Fenchel-Nielse coordinates, again leaving the proof to the reader. Note that the Dehn twist parameters are defined only in a relative way, however this is exactly the same as in the non-singular case (see e.g. [BP92]).

Corollary A.11. *Given a (topological) pants decomposition of S with boundary curves $\gamma_1, \cdots, \gamma_N$, there is a homeomorphism*

$$\mathcal{T}_{S,x,\theta} \to (\mathbb{R}_{>0} \times \mathbb{R})^N$$

sending a hyperbolic metric to the length and fractional Dehn twist parameters at the γ_i.

The fractional Dehn twist parameters used here are the translation length of one side with respect to the other so that, for a boundary curve of length l, a parameter equal to l corresponds to a "usual" Dehn twist (the other possibility is to use an "angle" parameter, where 2π corresponds to full Dehn twist).

A.2 Proof of Proposition 1.15

It is now possible to use the pants decomposition provided by Lemma A.3 to prove Proposition 1.15: the induced metric on the boundary of the convex core is (uniformly) quasi-conformal to the conformal structure at infinity.

The starting point is that a pants decomposition of $(\partial M, m)$ with boundary curves of bounded length defines a pants decomposition of $(\partial M, \tau)$ with boundary curves of approximately the same length. Recall that the constant C_p was introduced in Lemma A.3.

Lemma A.12. *There exists a constant $C > 0$ as follows. Let $\gamma_1, \cdots, \gamma_N$ be simple closed curves on ∂M, defining a pants decomposition, of lengths less than C_p for m. Then*

$$\forall i \in \{1, \cdots, N\}, \frac{L_m(\gamma_i)}{C} \le L_\tau(\gamma_i) \le C L_m(\gamma_i) \ .$$

Proof. The upper bound is a direct consequence of the first point in Proposition 5.12. If γ_i is short for m – i.e., it is the core of a long tube in the thin part of $(\partial M, m)$ – then the second point of Proposition 5.12 proves the lower bound for γ_i.

Suppose now that γ_i is realized in $(\partial M, m)$ as a closed geodesic in the thick part of ∂M. Then there exists a closed geodesic γ' intersecting γ_i of length at most C_p. If the length of γ_i in $(\partial M, \tau)$ were small, than γ_i would be realized in $(\partial M, \tau)$ as the core of a long tube T in the thin part of $(\partial M, \tau)$. But then γ' would have to be long (at least as long as the T). This would contradict the first point in Proposition 5.12, and this proves the lower bound for γ_i. □

Lemma A.13. *There exists a constant $C > 0$ such that, for each of the γ_i, the difference in the Dehn twist parameter corresponding to γ_i in m and in τ is at most $C(|\log(L_m(\gamma_i))| + 1)$.*

The precise form of the estimate is important only if γ_i is short for m (and therefore for τ), in which case $|\log(L_m(\gamma_i))|$ is half the length of the tube in the thin part of $(\partial M, m)$ containing γ_i.

Proof. Suppose first that γ_i is not short. Then it is contained in the thick part of $(\partial M, m)$, and there exists another curve γ', intersecting γ_i, of uniformly bounded length. A Dehn twist parameter bigger than some constant would extend the length of γ' by more than is allowed by Proposition 5.12, this proves the lemma in this first case.

The same argument can be used when γ_i is short (i.e. when it is the core of a long thin tube), then γ' can be chosen to have length bounded by a constant time $|\log(L_m(\gamma_i))|$, and this defines the maximal Dehn twist parameter along γ_i. □

Proof of Proposition 1.15. Let $\gamma_1, \cdots, \gamma_N$ be disjoint closed curves, defining a pants decomposition of $(\partial M, m)$ with boundary curves of length less than C_p, as in Lemma A.3. Let $l_1, \cdots, l_N$ be the length of the γ_i for m, and let $d_1, \cdots, d_N$ be the Dehn twist parameters for the same curves.

Lemma A.10 shows that the γ_i also define a pant decomposition of $(\partial M, \tau)$, let l'_i be the length of the γ_i for τ, and let d'_i be their Dehn twist parameters. Lemma A.12 indicates that the l'_i are within a fixed multiplicative constant from the l_i, while, by Lemma A.13,

$$|d'_i - d_i| \le C(|\log(l_i)| + 1) \ , \tag{3}$$

where C is some positive constant.

Let m' be the hyperbolic metric with cone singularities obtained by gluing pairs of pants with boundary lengths equal to the l_i, but with Dehn twist parameters equal to the d'_i.

Note that m' is C_1-quasi-conformal to m, for some uniform constant $C_1 > 0$. To prove this remark that for each $i \in \{1, \cdots, N\}$ the set of points at distance at most $C(|\log(L_m(\gamma_i))| + c_M)$ from γ_i is an annulus, and that those annuli are disjoint. One can therefore built a C_1-quasi-conformal diffeomorphism between m and m' which is an isometry in the complement of those annuli around the γ_i, and which is "twisted" in those annuli, with a twisting parameter which is an affine function of the distance to the γ_i.

The second and last step is that m' is C_3-quasi-conformal τ. Since those two metrics differ only by the lengths of the boundary curves γ_i, and in view of (3), this follows again from a simple and explicit construction which we leave to the interested reader. □

Acknowledgements

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